CATEGORIFYING THE TENSOR PRODUCT OF A LEVEL 1 HIGHEST WEIGHT AND PERFECT CRYSTAL IN TYPE A

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Abstract. We use KLR algebras to categorify a crystal isomorphism between a highest weight crystal and the tensor product of a perfect crystal and another highest weight crystal, all in level 1 type A affine. The nodes of the perfect crystal correspond to a family of trivial modules and the nodes of the highest weight crystal correspond to simple modules, which we may also parameterize by ℓ-restricted partitions. In the case ℓ is a prime, one can reinterpret all the results for the symmetric group in characteristic ℓ. The crystal operators correspond to socle of restriction and behave compatibly with the rule for tensor product of crystal graphs.

Contents

1. Introduction

Kang-Kashiwara [10] and Webster [28] show the cyclotomic Khovanov-Lauda-Rouquier (KLR) algebra R^Λ categorifies the highest weight representation $V(\Lambda)$ in arbitrary symmetrizable type. (KLR algebras are also known as quiver Hecke algebras.) We will say the *combinatorial* version of this statement is that R^Λ categorifies the crystal $B(\Lambda)$, where simple modules correspond to

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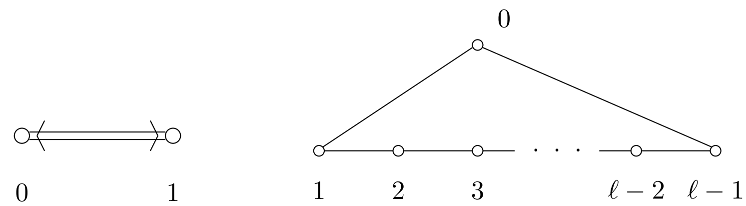

FIGURE 1. The Dynkin diagram for $A_1^{(1)}$ is on the left and and the Dynkin diagram for $A_{\ell-1}^{(1)}$ with $\ell > 2$ is on the right.

nodes, and functors that take socle of restriction correspond to arrows, i.e. the Kashiwara crystal operators. Webster [28] and Losev-Webster [22] categorify the tensor product of highest weight modules, and hence the tensor product of highest weight crystals. However, one can consider a tensor product of crystals

$$\mathcal{B} \otimes B(\Lambda) \simeq B(\Lambda') \tag{1.1}$$

where $\Lambda, \Lambda' \in P^+$ are of level k and $\mathcal{B}$ is a perfect crystal of level k. In this paper, we (combinatorially) categorify the crystal isomorphism (1.1) in the case the level $k = 1$ for type $A_{\ell-1}^{(1)}$ and $\mathcal{B} = B^{1,1}$ which is drawn in Figure 2. Each node of $\mathcal{B}$ corresponds to a family of trivial modules, but note this does *not* give a categorification of $\mathcal{B}$. (By symmetry we have similar results for $\mathcal{B} = B^{\ell-1,1}$ of Figure 5 whose nodes correspond to sign modules.)

We note that this gives a construction of simple modules that is somewhat intermediate between the crystal operator construction and the Specht module construction. Combinatorially, the former corresponds to building an ℓ-restricted partition one (good) box at a time. Our construction builds a partition one row at a time, or dually one column at a time. The Specht module construction (at least for $\mathbb{F}_\ell \mathcal{S}_n$ or the Hecke algebra of type A) builds the simple from the whole partition, constructing the simple as a subquotient of an induced trivial module from a parabolic subalgebra that corresponds to the partition. However, this paper also describes how socle of restriction interacts with the construction. One can also recover this construction for finite type $A_{\ell-1}$ as its Dynkin diagram is a subdiagram of that of type $A_{\ell-1}^{(1)}$, or recovers characteristic 0 constructions taking $\ell \to \infty$. For a construction of simple modules related to the crystal $B(\infty)$ for finite type KLR algebras see [2].

This paper is based on unpublished work of the author [26, 25] which was done for the affine Hecke algebra of type A at an ℓth root of unity. We chose to rewrite this in the language of KLR algebras to appeal to the modern reader and also make it easier to then generalize the theorem to other affine types in [20].

I wish to thank Henry Kvinge for his help with the figures and whose feedback greatly improved the exposition.

2. Type A Cartan datum and crystals

2.1. Cartan datum for type $A_{\ell-1}^{(1)}$. Fix an integer $\ell \geq 2$. In this paper we will work solely in type $A_{\ell-1}^{(1)}$. Let I be the indexing set

$$I = \{0, 1, \ldots, \ell - 1\}. \tag{2.1}$$

Let $[a_{ij}]_{i,j\in I}$ denote the associated Cartan matrix. For $\ell > 2$ the type A Cartan matrix is the $\ell \times \ell$ matrix

$$\begin{bmatrix} 2 & -1 & 0 & & \cdots & 0 & 0 \\ -1 & 2 & -1 & & \cdots & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & & \cdots & -1 & 2 & -1 \\ 0 & 0 & & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

When $\ell = 2$ it is

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

Following [9] we let $\mathfrak{h}$ be a Cartan subalgebra, $\prod = \{\alpha_0, \dots, \alpha_{\ell-1}\}$ its system of simple roots, $\prod^\vee = \{h_0, \dots, h_{\ell-1}\}$ its simple coroots, and Q and $Q^\vee$ the root and coroot lattices respectively. Then set

$$Q^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i. \tag{2.2}$$

For an element $\nu \in Q^+$, we define its *height*, $|\nu|$, to be the sum of the coefficients, i.e. if $\nu = \sum_{i\in I} \nu_i\alpha_i$ then

$$|\nu| = \sum_{i\in I} \nu_i. \tag{2.3}$$

We also have a symmetric bilinear form

$$(\ ,\ ) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$$

which satisfies

$$a_{ij} = \langle h_i, \alpha_j\rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \tag{2.4}$$

where $\langle\ ,\ \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ is the canonical pairing. Using this pairing we define the fundamental weights $\{\Lambda_i \mid i \in I\}$ via

$$\langle h_j, \Lambda_i\rangle = \delta_{ji}.$$

The weight lattice is $\bigoplus_{i\in I} \mathbb{Z}\Lambda_i$ and the integral dominant weights are

$$P^+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\Lambda_i.$$

Remark 2.1. Because in this paper we work exclusively with Cartan datum associated with $A^{(1)}_{\ell-1}$, it is often convenient to identify elements of I with $\mathbb{Z}/\ell\mathbb{Z}$, so when stating that $k \in I$, we will usually think of $k \in \mathbb{Z}/\ell\mathbb{Z}$ even if we neglect to write $\overline{k}$ or $k \bmod \ell$. We will often be considering a sequence of k operators, $k \in \mathbb{N} := \mathbb{Z}_{\geq 0}$, but the kth operator may be indexed by $(k-1) \bmod \ell$, for which it is convenient to relax notation.

2.2. **Review of crystals.** We recall the tensor category of crystals following Kashiwara [14], see also [13, 12, 15].

A *crystal* is a set B together with maps

- $\mathrm{wt}\colon B \longrightarrow P$,
- $\varepsilon_i, \varphi_i : B \longrightarrow \mathbb{Z} \sqcup \{-\infty\}$ for $i \in I$,
- $\widetilde{e}_i, \widetilde{f}_i\colon B \longrightarrow B \sqcup \{0\}$ for $i \in I$,

such that

C1. $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for any i.

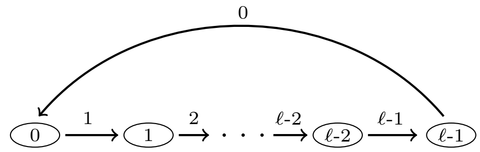

FIGURE 2. The level 1 perfect crystal $\mathcal{B}$, which is also denoted $B^{1,1}$.

C2. If $b \in B$ satisfies $\widetilde{e}_i b \neq 0$, then

$$\varepsilon_i(\widetilde{e}_i b) = \varepsilon_i(b) - 1, \qquad \varphi_i(\widetilde{e}_i b) = \varphi_i(b) + 1, \qquad \mathrm{wt}(\widetilde{e}_i b) = \mathrm{wt}(b) + \alpha_i.$$

C3. If $b \in B$ satisfies $\widetilde{f}_i b \neq 0$, then

$$\varepsilon_i(\widetilde{f}_i b) = \varepsilon_i(b) + 1, \qquad \varphi_i(\widetilde{f}_i b) = \varphi_i(b) - 1, \qquad \mathrm{wt}(\widetilde{f}_i b) = \mathrm{wt}(b) - \alpha_i.$$

C4. For b_1, $b_2 \in B$, $b_2 = \widetilde{f}_i b_1$ if and only if $\widetilde{e}_i b_2 = b_1$.
C5. If $\varphi_i(b) = -\infty$, then $\widetilde{e}_i b = \widetilde{f}_i b = 0$.

If B_1 and B_2 are two crystals, then a *morphism* $\psi\colon B_1 \to B_2$ of crystals is a map

$$\psi\colon B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$$

satisfying the following properties:
M1. $\psi(0) = 0$.
M2. If $\psi(b) \neq 0$ for $b \in B_1$, then

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b), \qquad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \qquad \varphi_i(\psi(b)) = \varphi_i(b).$$

M3. For $b \in B_1$ such that $\psi(b) \neq 0$ and $\psi(\widetilde{e}_i b) \neq 0$, we have $\psi(\widetilde{e}_i b) = \widetilde{e}_i(\psi(b))$.
M4. For $b \in B_1$ such that $\psi(b) \neq 0$ and $\psi(\widetilde{f}_i b) \neq 0$, we have $\psi(\widetilde{f}_i b) = \widetilde{f}_i(\psi(b))$.
A morphism ψ of crystals is called *strict* if

$$\psi \circ \widetilde{e}_i = \widetilde{e}_i \circ \psi, \qquad \psi \circ \widetilde{f}_i = \widetilde{f}_i \circ \psi,$$

and an *embedding* if ψ is injective.

Given two crystals B_1 and B_2 their tensor product $B_1 \otimes B_2$ (using the reverse Kashiwara convention) has underlying set $\{b_1 \otimes b_2; b_1 \in B_1, \text{ and } b_2 \in B_2\}$ where we identify $b_1 \otimes 0 = 0 \otimes b_2 = 0$. The crystal structure is given as follows:

$$\mathrm{wt}(b_1 \otimes b_2) = \mathrm{wt}(b_1) + \mathrm{wt}(b_2), \tag{2.5}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max\{\varepsilon_i(b_2), \varepsilon_i(b_1) - \langle h_i, \mathrm{wt}(b_2)\rangle\}, \tag{2.6}$$

$$\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_2) + \langle h_i, \mathrm{wt}(b_1)\rangle, \varphi_i(b_1)\}, \tag{2.7}$$

$$\widetilde{e}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2) \\ b_1 \otimes \widetilde{e}_i b_2 & \text{if } \varepsilon_i(b_1) \leq \varphi_i(b_2), \end{cases} \tag{2.8}$$

$$\widetilde{f}_i(b_1 \otimes b_2) = \begin{cases} \widetilde{f}_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) \geq \varphi_i(b_2) \\ b_1 \otimes \widetilde{f}_i b_2 & \text{if } \varepsilon_i(b_1) < \varphi_i(b_2). \end{cases} \tag{2.9}$$

Given a crystal B, we can draw its associated crystal graph with nodes (or vertices) B and I-colored arrows (directed edges) as follows. When $\widetilde{e}_i b = a$ (so $b = \widetilde{f}_i a$) we draw an i-colored arrow $a \xrightarrow{i} b$. We also say b has an incoming i-arrow and a has an outgoing i-arrow.

3. Level 1 crystals in type $A^{(1)}_{\ell-1}$

The level 1 highest weight crystal, or fundamental crystal, $B(\Lambda_i)$ has a model (see Figure 3) with nodes *ℓ-restricted* partitions, i.e. $\lambda = (\lambda_1, \dots, \lambda_t)$ such that $\lambda_r \in \mathbb{Z}_{\geq 0}$, $0 \leq \lambda_r - \lambda_{r+1} < \ell$ for all r. Observe that for fixed ℓ, as directed graphs $B(\Lambda_0)$ and $B(\Lambda_i)$ are identical. The edge labels or "colors" for $B(\Lambda_i)$ are obtained from those of $B(\Lambda_0)$ by adding i mod ℓ.

Let $\mathcal{B}$ be the crystal graph in Figure 2. $\mathcal{B}$ is an example of a level 1 *perfect* crystal. See [11] for the definition of a perfect crystal and for many of its important properties. $\mathcal{B}$, often denoted $B^{1,1}$ in the literature is also an example of a Kirillov-Reshetikhin crystal. Observe that we have parameterized the nodes of $\mathcal{B}$ so that

$$\varepsilon_i(k) = \delta_{i,k}.$$

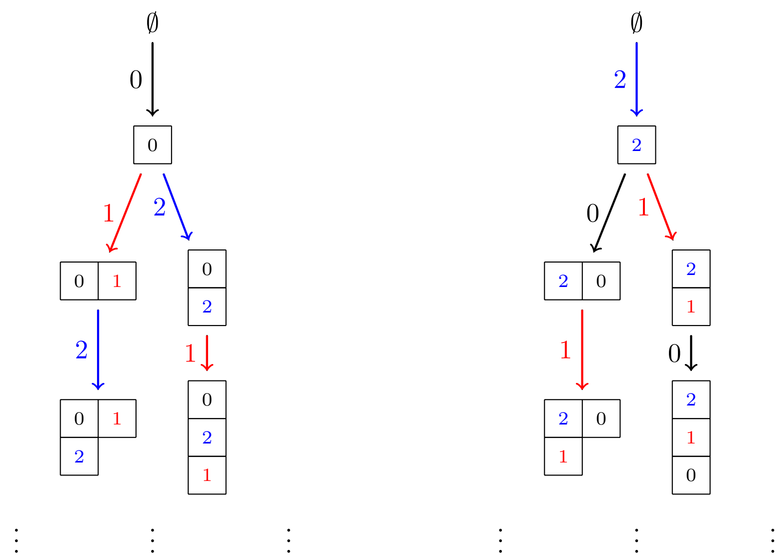

Figure 3. $B(\Lambda_0)$ and $B(\Lambda_2)$ for $\ell = 3$.

Then $\mathcal{T} : B(\Lambda_i) \xrightarrow{\simeq} \mathcal{B} \otimes B(\Lambda_{i-1})$ is an isomorphism of crystals. The isomorphism is pictured in Figure 4 for $i = 0$ and $\ell = 3$. Combinatorially, $\mathcal{T}(\lambda) = k \otimes \mu$ where $k \equiv \lambda_1 + i - 1 \bmod \ell$ and $\mu = (\lambda_2, \dots, \lambda_t)$ if $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$. So we obtain μ from λ by removing its top row. In Figure 4, we draw

$$\mathcal{T}(\lambda) = \begin{matrix} k \\ \otimes \\ \mu \end{matrix} \tag{3.1}$$

so the visual of the top row removal stands out. Note $\lambda_1 - \mu_1 = \lambda_1 - \lambda_2 < \ell$ means that $\mathcal{T}$ has a well-defined inverse.

When drawing our model of $B(\Lambda_i)$, we label each box of an ℓ-restricted partition with $k \in I$, such that the main diagonal gets label i, and labels increase by 1 mod ℓ as one increases diagonals (moving right). In this manner, the last box in the top row of λ is labeled k when $\mathcal{T}(\lambda) = k \otimes \mu$. Note further that if we have a k-arrow $\gamma \xrightarrow{k} \lambda$ then the box λ/γ is labeled k (though not necessarily conversely). In fact, once one knows the structure of $\mathcal{B}$ and the tensor product rule for crystals, one can obtain the rule for which k-box $\widetilde{e}_k$ removes by iterating $\mathcal{T}$.

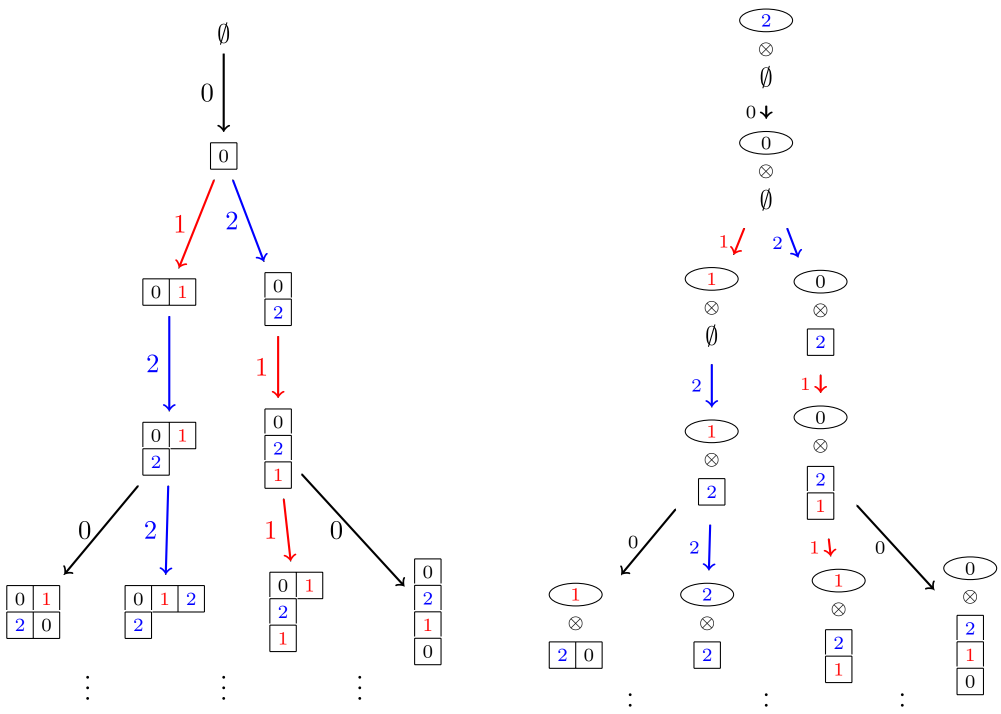

FIGURE 4. The isomorphism $B(\Lambda_0) \simeq \mathcal{B} \otimes B(\Lambda_2)$ for $\ell = 3$.

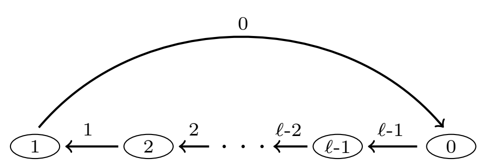

FIGURE 5. The level 1 perfect crystal $\mathcal{B}^{\text{opp}}$, which is also denoted $B^{\ell-1,1}$.

$\mathcal{B}^{\text{opp}}$, often denoted $B^{\ell-1,1}$ in the literature, is another level 1 perfect crystal and is also an example of a Kirillov-Reshetikhin crystal. $\mathcal{B}^{\text{opp}}$ is pictured in Figure 5; note it can be obtained from $\mathcal{B}$ from reversing orientation of all arrows, and we chose to relabel nodes so that still

$$\varepsilon_i(k) = \delta_{i,k}.$$

We have another crystal isomorphism $\mathcal{T}^{\text{opp}} : B(\Lambda_i) \xrightarrow{\simeq} \mathcal{B}^{\text{opp}} \otimes B(\Lambda_{i+1})$. See Figure 6. This isomorphism is compatible with the model of $B(\Lambda_i)$ that labels nodes with ℓ-*regular* partitions, that is, those partitions μ such that the transposed diagram μ^T is ℓ-restricted. Then the isomorphism $\mathcal{T}^{\text{opp}}$ corresponds to column removal, in the same way $\mathcal{T}$ corresponds to row removal.

While the underlying I-colored directed graphs $B(\Lambda_i)$ are identical, one does not obtain the ℓ-regular model by merely transposing the partition indexing each node of the ℓ-restricted model. See Section 5.0.1 for another model of $B(\Lambda_i)$ that comes from KLR algebras.

There are other level 1 perfect crystals besides $\mathcal{B}$ and $\mathcal{B}^{\text{opp}}$, but we do not consider them here.

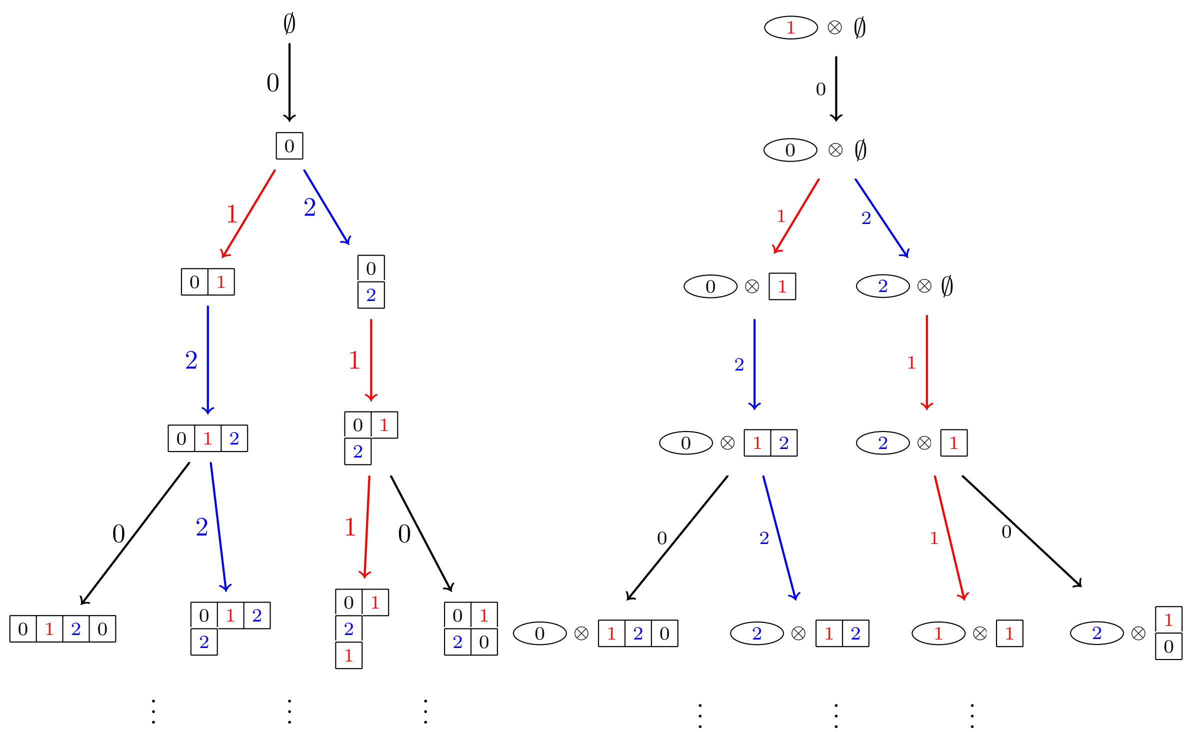

FIGURE 6. The isomorphism $B(\Lambda_0) \simeq \mathcal{B}^{\mathrm{opp}} \otimes B(\Lambda_1)$ for $\ell = 3$.

4. Definition of the KLR algebra $R(\nu)$ and some functors

In what follows we let $[k]$ be the quantum integer in the indeterminant q,

$$[k] = q^{k-1} + q^{k-3} + \cdots + q^{1-k} \quad \text{and} \quad [k]! = [k][k-1]\ldots[1]. \tag{4.1}$$

For $\nu = \sum_{i\in I} \nu_i\alpha_i$ in Q^+ with $|\nu| = m$, we define $\mathrm{Seq}(\nu)$ to be all sequences

$$\underline{\mathbf{i}} = (i_1, i_2, \ldots, i_m)$$

such that i_k appears ν_k times. For $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$ and $\underline{\mathbf{j}} \in \mathrm{Seq}(\mu)$, $\underline{\mathbf{ij}}$, will denote the concatenation of the two sequences unless otherwise specified. It follows that $\underline{\mathbf{ij}} \in \mathrm{Seq}(\nu + \mu)$. We write

$$i^n = (\underbrace{i, i, \ldots, i}_{n}). \tag{4.2}$$

There is a left action of the symmetric group, $\mathcal{S}_m$, on $\mathrm{Seq}(\nu)$ defined by,

$$s_k(\underline{\mathbf{i}}) = s_k\Big(i_1, i_2, \ldots, i_k, i_{k+1}, \ldots, i_m\Big) = (i_1, i_2, \ldots, i_{k+1}, i_k, \ldots, i_m) \tag{4.3}$$

where s_k is the adjacent transposition in $\mathcal{S}_m$ that interchanges k and $k+1$.

Since this paper only considers KLR algebras of type $A^{(1)}_{\ell-1}$, we simplify the definition below from that for general type. The definition for arbitrary symmetrizable types can be found in [16], [17], and [24]. Using the more general definition with Rouquier's parameters $Q_{i,j}(u, v)$ will not change the results or proofs in this paper, as they concern crystal-theoretic phenomena. There is also a diagrammatic presentation of KLR algebras which can be found in [16], [17]. By results of Brundan-Kleshchev [4], [5], there is an isomorphism between $R^\Lambda(\nu)$ and H^Λ_ν where H^Λ_ν is a block of the cyclotomic Hecke algebra H^Λ_m as defined in [1, 3, 6]. Hence readers unfamiliar with KLR

algebras can translate all statements and proofs in terms of Hecke algebras throughout the paper. We remark that historically, this is the original setting in which the theorems from this paper were proved [26, 25]. In fact the reader can think of all results as being stated for $\mathbb{F}_\ell \mathcal{S}_m$ in the case that ℓ is prime if the other algebras are not familiar.

For $\nu \in Q^+$ with $|\nu| = m$, the *KLR algebra* $R(\nu)$ is the associative, graded, unital $\mathbb{C}$-algebra generated by

$$1_{\underline{\mathbf{i}}} \text{ for } \underline{\mathbf{i}} \in \mathrm{Seq}(\nu), \quad x_r \text{ for } 1 \le r \le m, \quad \psi_r \text{ for } 1 \le r \le m-1, \tag{4.4}$$

subject to the following relations, where $\underline{\mathbf{i}}, \underline{\mathbf{j}} \in \mathrm{Seq}(\nu)$ and equality between i_r and i_t is taken to mean equality in $\mathbb{Z}/\ell\mathbb{Z}$.

$$1_{\underline{\mathbf{i}}} 1_{\underline{\mathbf{j}}} = \delta_{\underline{\mathbf{i}},\underline{\mathbf{j}}} 1_{\underline{\mathbf{i}}}, \qquad x_r 1_{\underline{\mathbf{i}}} = 1_{\underline{\mathbf{i}}} x_r, \qquad \psi_r 1_{\underline{\mathbf{i}}} = 1_{s_r(\underline{\mathbf{i}})} \psi_r, \qquad x_r x_t = x_t x_r, \tag{4.5}$$

$$\psi_r \psi_t = \psi_t \psi_r \qquad \text{if } |r-t| > 1, \tag{4.6}$$

$$\psi_r \psi_r 1_{\underline{\mathbf{i}}} = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ (x_r^{-a_{i_r i_{r+1}}} + x_{r+1}^{-a_{i_{r+1} i_r}}) 1_{\underline{\mathbf{i}}} & \text{if } i_r = i_{r+1} \pm 1 \\ 1_{\underline{\mathbf{i}}} & \text{otherwise,} \end{cases} \tag{4.7}$$

$$(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) 1_{\underline{\mathbf{i}}} = \begin{cases} 1_{\underline{\mathbf{i}}} & \text{if } \ell > 2 \text{ and } i_r = i_{r+2} = i_{r+1} \pm 1 \\ (x_r + x_{r+2}) 1_{\underline{\mathbf{i}}} & \text{if } \ell = 2 \text{ and } i_r = i_{r+2} = i_{r+1} \pm 1 \\ 0 & \text{otherwise,} \end{cases} \tag{4.8}$$

$$(\psi_r x_t - x_{s_r(t)} \psi_r) 1_{\underline{\mathbf{i}}} = \begin{cases} 1_{\underline{\mathbf{i}}} & \text{if } t = r \text{ and } i_r = i_{r+1} \\ -1_{\underline{\mathbf{i}}} & \text{if } t = r+1 \text{ and } i_r = i_{r+1} \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

The elements $1_{\underline{\mathbf{i}}}$ are idempotents in $R(\nu)$ by (4.5) and the identity element is given by

$$1_\nu = \sum_{\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)} 1_{\underline{\mathbf{i}}}. \tag{4.10}$$

Thus, as a vector space $R(\nu)$ decomposes as,

$$R(\nu) = \bigoplus_{\underline{\mathbf{i}},\underline{\mathbf{j}} \in \mathrm{Seq}(\nu)} 1_{\underline{\mathbf{i}}} R(\nu) 1_{\underline{\mathbf{j}}}. \tag{4.11}$$

The generators of $R(\nu)$ are graded as,

$$\deg(1_{\underline{\mathbf{i}}}) = 0, \quad \deg(x_r 1_{\underline{\mathbf{i}}}) = 2, \quad \deg(\psi_r 1_{\underline{\mathbf{i}}}) = -(\alpha_{i_r}, \alpha_{i_{r+1}}). \tag{4.12}$$

We define

$$R = \bigoplus_{\nu \in Q^+} R(\nu). \tag{4.13}$$

Notice that while $R(\nu)$ is unital, R is not.

For each $w \in \mathcal{S}_m$ we fix once and for all a reduced expression

$$\widehat{w} = s_{i_1} s_{i_2} \dots s_{i_t}. \tag{4.14}$$

Observe that the s_k are Coxeter generators of $\mathcal{S}_m$, and t is the Coxeter length of w. Let $\psi_{\widehat{w}} = \psi_{i_1} \psi_{i_2} \dots \psi_{i_t}$ correspond to the chosen reduced expression $\widehat{w}$. For $\underline{\mathbf{i}}, \underline{\mathbf{j}} \in \mathrm{Seq}(\nu)$, let ${}_{\underline{\mathbf{j}}}\mathcal{S}_{\underline{\mathbf{i}}}$ be the permutations in $\mathcal{S}_m$ that take $\underline{\mathbf{i}}$ to $\underline{\mathbf{j}}$.

Theorem 4.1. [16, Theorem 2.5] *As a* $\mathbb{C}$*-vector space* $1_{\underline{\mathbf{j}}} R(\nu) 1_{\underline{\mathbf{i}}}$ *has basis,*

$$\{\psi_{\widehat{w}} x_1^{b_1} \dots x_m^{b_m} 1_{\underline{\mathbf{i}}} \mid w \in {}_{\underline{\mathbf{j}}}\mathcal{S}_{\underline{\mathbf{i}}},\ b_r \in \mathbb{Z}_{\ge 0}\}. \tag{4.15}$$

It is known that all simple $R(\nu)$-modules are finite dimensional [16]. For this reason, in this paper we only consider the category of finite-dimensional KLR-modules $R(\nu)\text{-}\mathrm{mod}$ and $R\text{-}\mathrm{mod}$.

We often refer to $1_{\underline{\mathbf{i}}}M$ as the $\underline{\mathbf{i}}$-*weight* space of M and any $0 \neq v \in 1_{\underline{\mathbf{i}}}M$ as a *weight vector*. A *weight basis* is a basis consisting of weight vectors.

We define the graded character of an $R(\nu)$-module to be

$$\mathrm{Char}(M) = \sum_{\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)} \mathrm{gdim}(1_{\underline{\mathbf{i}}}M) \cdot [\underline{\mathbf{i}}]. \tag{4.16}$$

Here $\mathrm{gdim}(1_{\underline{\mathbf{i}}}M)$ is an element of $\mathbb{Z}[q, q^{-1}]$, and hence $\mathrm{Char}(M)$ is an element of the free $\mathbb{Z}[q, q^{-1}]$-module generated by all $[\underline{\mathbf{i}}]$ for $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$. We will let $\mathrm{supp}(M)$ denote the multiset that is the support of $\mathrm{Char}(M)$ so that

$$\mathrm{Char}(M)|_{q=1} = \sum_{[\underline{\mathbf{i}}] \in \mathrm{supp}(M)} [\underline{\mathbf{i}}]. \tag{4.17}$$

Our notational convention is to write $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$ but write $[\underline{\mathbf{i}}] \in \mathrm{supp}(M)$. Since characters are an important combinatorial tool, it is worthwhile to set a special notation for them.

Because R is a graded algebra, we will only work with homomorphisms between R-modules that are either degree preserving or degree homogeneous. We denote the $\mathbb{C}$-vector space of degree preserving homomorphisms between $R(\nu)$-modules M and N by $\mathrm{Hom}(M, N)$. Since any homogeneous homomorphism can be interpreted as degree preserving by shifting the grading on our target or source module, then we can write the $\mathbb{C}$-vector space of homogeneous homomorphisms between M and N, $\mathrm{HOM}(M, N)$, by

$$\mathrm{HOM}(M, N) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}(M, N\{k\}). \tag{4.18}$$

While the grading is important, it was shown in [16] that there is a unique grading on a simple R-module up to overall grading shift. Since this paper concerns simple modules, we will rarely use or discuss the grading. All isomorphisms between modules will be taken up to overall grading shift.

Remark 4.2. Because $x_r 1_{\underline{\mathbf{i}}} \in R(\nu)$ is always positively graded for $1 \leq r \leq |\nu|$ and $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$, then on a finite dimensional $R(\nu)$-module, M, $x_r 1_{\underline{\mathbf{i}}}$ will always act nilpotently.

4.1. **Trivial and sign modules.** For $k \in \mathbb{N}, i \in I$, define positive roots

$$\gamma^+_{i;k} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{i+k-1} \quad \text{and} \quad \gamma^-_{i;k} := \alpha_i + \alpha_{i-1} + \cdots + \alpha_{i-k+1} \tag{4.19}$$

of height k. As in Remark 2.1 we interpret subscripts to be in I. Because 1-dimensional modules will play a key role in our main theorems below, we give the following classification.

Proposition 4.3. If M is a 1-dimensional $R(\nu)$-module with $|\nu| = m$, then M has character

$$\mathrm{Char}(M) = [i, i+1, \ldots, i+m-2, i+m-1] \tag{4.20}$$

or

$$\mathrm{Char}(M) = [i, i-1, \ldots, i-m+2, i-m+1] \tag{4.21}$$

and $\nu = \gamma^+_{i;m}$ or $\nu = \gamma^-_{i;m}$ respectively. The entries in $\mathrm{Char}(M)$ should be taken modulo ℓ.

Proof. Our proof uses similar techniques to those used in [19] for the study of calibrated (or homogeneous) modules, of which the 1-dimensional $R(\nu)$-modules form a subset. Let M be spanned by the vector v, and let $\underline{\mathbf{i}}$ be the unique element of $\mathrm{Seq}(\nu)$ such that $1_{\underline{\mathbf{i}}}M \neq \mathbf{0}$. We write

$$\underline{\mathbf{i}} = (i_1, i_2, \ldots, i_m). \tag{4.22}$$

Recall $x_r 1_{\underline{\mathbf{i}}}$, for $1 \leq r \leq m$ acts nilpotently on M by Remark 4.2. Since M is 1-dimensional, then $x_r 1_{\underline{\mathbf{i}}} v = 0$.

Suppose $i_r = i_{r+1}$ (recall that we interpret $i_r, i_{r+1} \in \mathbb{Z}/\ell\mathbb{Z}$). Then from relation (4.9),

$$(\psi_r x_r - x_{r+1}\psi_r)1_{\underline{\mathbf{i}}}v = 1_{\underline{\mathbf{i}}}v = v, \tag{4.23}$$

but this is impossible as x_r and x_{r+1} both act as zero. Thus $i_r \neq i_{r+1}$.

Let $\psi_r 1_{\underline{\mathbf{i}}}v = a_r v$ for some constant a_r. Given we showed $i_r \neq i_{r+1}$, if additionally we suppose $i_{r+1} \neq \pm 1 + i_r$, then by relation (4.7),

$$a_r^2 v = \psi_r^2 1_{\underline{\mathbf{i}}}v = v, \tag{4.24}$$

so $a_r \neq 0$. Then, $0 \neq \psi_r 1_{\underline{\mathbf{i}}}v = 1_{s_r(\underline{\mathbf{i}})}\psi_r v \in 1_{s_r(\underline{\mathbf{i}})}M$. But $s_r(\underline{\mathbf{i}}) \neq \underline{\mathbf{i}}$ given $i_r \neq i_{r+1}$ which contradicts the fact that M is 1-dimensional. Hence $i_{r+1} = \pm 1 + i_r$ and

$$a_r^2 v = \psi_r^2 1_{\underline{\mathbf{i}}}v = (x_r + x_{r+1})1_{\underline{\mathbf{i}}}v = 0 \tag{4.25}$$

showing $a_r = 0$. Thus $\psi_r 1_{\underline{\mathbf{i}}}v = 0$ for all r.

In the case when $\ell = 2$, $i_{r+1} = \pm 1 + i_r$ fully determines $\underline{\mathbf{i}}$ and agrees with the conclusions of the proposition, so for the rest of the proof we assume $\ell > 2$. Suppose $i_r = i_{r+2}$ for some $1 \leq r \leq m-2$. Since $i_{r+1} = \pm 1 + i_r = \pm 1 + i_{r+2}$, relation (4.8) gives

$$0 = (a_r^2 a_{r+1} - a_r a_{r+1}^2)1_{\underline{\mathbf{i}}}v = (\psi_r\psi_{r+1}\psi_r - \psi_{r+1}\psi_r\psi_{r+1})1_{\underline{\mathbf{i}}}v = 1_{\underline{\mathbf{i}}}v = v \tag{4.26}$$

which is a contradiction as $a_r = 0$ but $v \neq 0$. So $i_r \neq i_{r+2}$ and $\underline{\mathbf{i}}$ has form (4.20) or (4.21). In particular $\nu = \gamma_{i;m}^{\pm}$. To show that such M actually exist one need only check that setting $1_{\underline{\mathbf{i}}}v = v$ for $\underline{\mathbf{i}}$ as in (4.20) or (4.21), and $x_r v = \psi_r v = 1_{\underline{\mathbf{j}}}v = 0$ for $\underline{\mathbf{j}} \neq \underline{\mathbf{i}}$, satisfies all of the relations on the generators of $R(\gamma_{i;m}^{\pm})$.

□

When $k > 0$, we denote the 1-dimensional $R(\gamma_{i;k}^{+})$-module T with ascending character,

$$\operatorname{Char} T = [i, i+1, \ldots, i+k-1] \quad \text{as} \quad T = T_{i;k} = T(i, i+1, \ldots, i+k-1), \tag{4.27}$$

and the 1-dimensional $R(\gamma_{i;k}^{-})$-module S with descending character,

$$\operatorname{Char} S = [i, i-1, \ldots, i-k+1] \quad \text{as} \quad S = S_{i;k} = S(i, i-1, \ldots, i-k+1). \tag{4.28}$$

We refer to k as the *height* of $T_{i;k}$ and $S_{i;k}$ respectively. When $k = 0$, then $\gamma_{i;k}^{\pm} = 0$ and $T_{i;k} = \mathbb{1}$, the unique simple $R(0)$-module, which we will refer to as the *unit* module. In this paper we choose to work with 1-dimensional modules with ascending character. (These are analogous to trivial modules for the affine Hecke algebra or symmetric group.) One could also have chosen to use the 1-dimensional modules with descending character (analogous to sign modules) with obvious modifications. Hence we will informally refer to each type of module as a trivial or sign module, respectively.

Example 4.4. In type $A_3^{(1)}$, with

$$\nu = 2\alpha_0 + 2\alpha_1 + \alpha_2 + \alpha_3,$$

of height 6, $R(\nu)$ has a simple 1-dimensional module $T = T_{0;6}$ with

$$\operatorname{Char}(T) = [0, 1, 2, 3, 0, 1],$$

but it is convenient to also write this as

$$\operatorname{Char}(T) = [0, 1, 2, 3, 4, 5].$$

4.2. **Induction and restriction.** It was shown in [16] and [17] that for $\nu, \mu \in Q^+$ there is a non-unital embedding

$$R(\nu) \otimes R(\mu) \hookrightarrow R(\nu + \mu). \tag{4.29}$$

This map sends the idempotent $1_{\underline{\mathbf{i}}} \otimes 1_{\underline{\mathbf{j}}}$ to $1_{\underline{\mathbf{ij}}}$. The identity $1_\nu \otimes 1_\mu$ of $R(\nu) \otimes R(\mu)$ has as its image

$$\sum_{\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)} \sum_{\underline{\mathbf{j}} \in \mathrm{Seq}(\mu)} 1_{\underline{\mathbf{ij}}}. \tag{4.30}$$

Using this embedding one can define induction and restriction functors,

$$\begin{aligned} \mathrm{Ind}_{\nu,\mu}^{\nu+\mu} : (R(\nu) \otimes R(\mu))\text{-}\mathrm{mod} &\to R(\nu+\mu)\text{-}\mathrm{mod} \\ M &\mapsto R(\nu+\mu) \otimes_{R(\nu)\otimes R(\mu)} M \end{aligned} \tag{4.31}$$

and

$$\mathrm{Res}_{\nu,\mu}^{\nu+\mu} : R(\nu+\mu)\text{-}\mathrm{mod} \to (R(\nu) \otimes R(\mu))\text{-}\mathrm{mod}\,. \tag{4.32}$$

In the future we will write $\mathrm{Ind}_{\nu,\mu}^{\nu+\mu} = \mathrm{Ind}$ and $\mathrm{Res}_{\nu,\mu}^{\nu+\mu} = \mathrm{Res}$ when the algebras are understood from the context. More generally we can extend this embedding to finite tensor products

$$R(\nu^{(1)}) \otimes R(\nu^{(2)}) \otimes \cdots \otimes R(\nu^{(k)}) \hookrightarrow R(\nu^{(1)} + \nu^{(2)} + \cdots + \nu^{(k)}). \tag{4.33}$$

We refer to the image of this embedding as a *parabolic subalgebra* and denote it by $R(\underline{\nu}) \subset R(\nu^{(1)} + \cdots + \nu^{(k)})$. We denote the image of the identity under this embedding as $1_{\underline{\nu}}$. It follows from Theorem 4.1 that $R(\nu^{(1)} + \nu^{(2)} + \cdots + \nu^{(k)})1_{\underline{\nu}}$ is a free right $R(\underline{\nu})$-module and $1_{\underline{\nu}}R(\nu^{(1)} + \nu^{(2)} + \cdots + \nu^{(k)})$ is a free left $R(\underline{\nu})$-module. Let $m_i = |\nu^{(i)}|$ and set

$$P = (m_1, \ldots, m_k) \quad \text{and} \quad \mathcal{S}_P = \mathcal{S}_{m_1} \times \mathcal{S}_{m_1} \times \cdots \times \mathcal{S}_{m_k}. \tag{4.34}$$

Let $\mathcal{S}_{m_1+\cdots+m_k}/\mathcal{S}_P$ be the collection of minimal length left coset representatives of $\mathcal{S}_P$ in $\mathcal{S}_{m_1+\cdots+m_k}$ and $\mathcal{S}_P\backslash\mathcal{S}_{m_1+\cdots+m_k}$ be the collection of minimal length right coset representatives of $\mathcal{S}_P$ in $\mathcal{S}_{m_1+\cdots+m_k}$. We construct a weight basis for an induced module as follows. If M is an $R(\underline{\nu})$-module with weight basis U then $\mathrm{Ind}_{\underline{\nu}}^{\nu^{(1)}+\cdots+\nu^{(k)}} M$ has weight basis

$$\{\psi_{\widehat{w}} \otimes u \mid w \in \mathcal{S}_{m_1+\cdots+m_k}/\mathcal{S}_P, u \in U\}. \tag{4.35}$$

Induction is left adjoint to restriction (a property known as Frobenius reciprocity),

$$\mathrm{HOM}_{R(\nu^{(1)}+\cdots+\nu^{(k)})}(\mathrm{Ind}_{\underline{\nu}}^{\nu^{(1)}+\cdots+\nu^{(k)}} M, N) \cong \mathrm{HOM}_{R(\underline{\nu})}(M, \mathrm{Res}_{\underline{\nu}}^{\nu^{(1)}+\cdots+\nu^{(k)}} N). \tag{4.36}$$

Given $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$ and $\underline{\mathbf{j}} \in \mathrm{Seq}(\mu)$, a shuffle of $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ is an element $\underline{\mathbf{k}}$ of $\mathrm{Seq}(\nu+\mu)$ such that $\underline{\mathbf{k}}$ has $\underline{\mathbf{i}}$ as a subsequence and $\underline{\mathbf{j}}$ as the complementary subsequence. We denote by $\underline{\mathbf{i}} \sqcup \underline{\mathbf{j}}$ the formal sum of all shuffles of $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$. The multi-set of all shuffles of $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ are in bijection with the minimal length left coset representatives $\mathcal{S}_{|\nu|+|\mu|}/\mathcal{S}_{|\nu|} \times \mathcal{S}_{|\mu|}$. Using the definition of degree from KLR algebras, we can associate to any shuffle a degree which we denote as $\deg(\underline{\mathbf{i}}, \underline{\mathbf{j}}, \underline{\mathbf{k}})$. Then the quantum shuffle of $\underline{\mathbf{i}}$ and $\underline{\mathbf{j}}$ is

$$\underline{\mathbf{i}} \mathbin{\text{ш}} \underline{\mathbf{j}} = \sum_{\sigma \in \mathcal{S}_{|\nu|+|\mu|}/\mathcal{S}_{|\nu|} \times \mathcal{S}_{|\mu|}} q^{\deg(\underline{\mathbf{i}}, \underline{\mathbf{j}}, \sigma(\underline{\mathbf{ij}}))} \sigma(\underline{\mathbf{ij}}), \tag{4.37}$$

so that $\underline{\mathbf{i}} \sqcup \underline{\mathbf{j}} = (\underline{\mathbf{i}} \mathbin{\text{ш}} \underline{\mathbf{j}})|_{q=1}$. Note that we will usually shuffle characters, hence we also write $[\underline{\mathbf{i}}] \mathbin{\text{ш}} [\underline{\mathbf{j}}]$. For an $R(\mu)$-module M and $R(\nu)$-module N it was shown in [16] that

$$\mathrm{Char}(\mathrm{Ind}_{\mu,\nu}^{\mu+\nu} M \boxtimes N) = \mathrm{Char}(M) \mathbin{\text{ш}} \mathrm{Char}(N). \tag{4.38}$$

This identity is referred to as the **Shuffle Lemma**.

4.3. **Simple modules of $R(n\alpha_i)$.** For $\nu = n\alpha_i$, induction allows for a particularly easy description of all simple $R(n\alpha_i)$-modules. Let $L(i)$ be the 1-dimensional $R(\alpha_i)$-module. (Note $x_1 1_i$ acts as zero.) Then the unique simple $R(n\alpha_i)$ module is

$$L(i^n) := \operatorname{Ind}^{n\alpha_i}_{\alpha_i,\alpha_i,\ldots,\alpha_i} L(i) \boxtimes \cdots \boxtimes L(i) \tag{4.39}$$

up to overall grading shift, which we may shift to have character

$$\operatorname{Char}(L(i^n)) = [n]![i,i,\ldots,i]. \tag{4.40}$$

4.4. **Crystal operators on the category R-mod.** In the previous section we defined induction and restriction for KLR algebras. Following the work of Grojnowski [8] where crystal operators were developed as functors on the category of modules over affine Hecke algebras of type A (or [18] for $\mathbb{F}_\ell \mathcal{S}_m$), the KLR analogues of crystal operators were introduced in [16], and further developed in [21], [10]. For each $i \in I$, if $M \in R(\nu)$-mod and $\nu - \alpha_i \in Q^+$, define the functor $\Delta_i : R(\nu)\text{-mod} \to R(\nu-\alpha_i) \otimes R(\alpha_i)\text{-mod}$ as the restriction

$$\Delta_i M := \operatorname{Res}^{\nu}_{\nu-\alpha_i,\alpha_i} M. \tag{4.41}$$

Note that this is equivalent to multiplying M by $1_{\nu-\alpha_i} \otimes 1_{\alpha_i}$. It is also sometimes useful to think of this functor as killing all weight spaces corresponding to elements of $\operatorname{Seq}(\nu)$ that do not end in i. If $\nu - \alpha_i \notin Q^+$ then $\Delta_i M = \mathbf{0}$. We similarly define

$$\Delta_{i^n} M := \operatorname{Res}^{\nu}_{\nu-n\alpha_i,n\alpha_i} M. \tag{4.42}$$

Next define the functor $e_i : R(\nu)\text{-mod} \to R(\nu-\alpha_i)\text{-mod}$ as the restriction,

$$e_i M := \operatorname{Res}^{R(\nu-\alpha_i)\otimes R(\alpha_i)}_{R(\nu-\alpha_i)} \Delta_i M \tag{4.43}$$

When M is simple, we can further refine this functor by setting

$$\widetilde{e}_i M := \operatorname{soc} e_i M. \tag{4.44}$$

We measure how many times we can apply $\widetilde{e}_i$ to a simple module M by

$$\varepsilon_i(M) := \max\{n \geq 0 \mid (\widetilde{e}_i)^n M \neq \mathbf{0}\ \}. \tag{4.45}$$

Let $\widetilde{f}_i : R(\nu)\text{-mod} \to R(\nu+\alpha_i)\text{-mod}$ be defined by

$$\widetilde{f}_i M := \operatorname{cosoc} \operatorname{Ind} M \boxtimes L(i). \tag{4.46}$$

We also set $\operatorname{wt}(M) = -\nu$ if $M \in R(\nu)$-mod, and $\varphi_i(M) = \varepsilon_i(M) - \langle h_i, \nu \rangle$. This data is all part of a crystal datum that defines the structure of the crystal graph $B(\infty)$ on the simple R-modules. See Section 5.0.1.

Some of the most important facts about $e_i, \widetilde{e}, \widetilde{f}_i$ stated in [16] are given in the following proposition.

Proposition 4.5. Let $i \in I$, $\nu \in Q^+$, $n \in \mathbb{Z}_{>0}$.

(1) Let $M \in R(\nu)$-mod. Then

$$\operatorname{Char}(\Delta_{i^n} M) = \sum_{\underline{\mathbf{j}} \in \operatorname{Seq}(\nu - n\alpha_i)} \operatorname{gdim}(1_{\underline{\mathbf{j}} i^n} M) \cdot \underline{\mathbf{j}} i^n,$$

(2) Let $N \in R(\nu)$-mod be irreducible and $M = \operatorname{Ind}^{\nu+n\alpha_i}_{\nu,n\alpha_i} N \boxtimes L(i^n)$. Let $\varepsilon = \varepsilon_i(N)$. Then
   (a) $\Delta_{i^{\varepsilon+n}} M \cong (\widetilde{e}_i)^\varepsilon N \boxtimes L(i^{\varepsilon+n})$.
   (b) $\operatorname{cosoc} M$ is irreducible, and $\operatorname{cosoc} M \cong (\widetilde{f}_i)^n N$, $\Delta_{i^{\varepsilon+n}} (\widetilde{f}_i)^n N \cong (\widetilde{e}_i)^\varepsilon N \boxtimes L(i^{\varepsilon+n})$, and $\varepsilon_i((\widetilde{f}_i)^n N) = \varepsilon + n$.
   (c) $(\widetilde{f}_i)^n N$ occurs with multiplicity one as a composition factor of M.
   (d) All other composition factors K of M have $\varepsilon_i(K) < \varepsilon + n$.

(3) Let $\underline{\mu} = (\mu_1\alpha_i, \ldots, \mu_r\alpha_i)$ with $\sum_{k=1}^r \mu_k = n$.

(a) All composition factors of $\mathrm{Res}^{n\alpha_i}_{\underline{\mu}} L(i^n)$ are isomorphic to $L(i^{\mu_1}) \boxtimes \cdots \boxtimes L(i^{\mu_r})$, and $\mathrm{soc}(\mathrm{Res}^{n\alpha_i}_{\underline{\mu}} L(i^n))$ is irreducible.

(b) $\widetilde{e}_i L(i^n) \cong L(i^{n-1})$.

(4) Let $M \in R(\nu)$-mod be irreducible with $\varepsilon_i(M) > 0$. Then $\widetilde{e}_i M = \mathrm{soc}(e_i M)$ is irreducible and $\varepsilon_i(\widetilde{e}_i M) = \varepsilon_i(M) - 1$. Furthermore if K is a composition factor of $e_i M$ and $K \ncong \widetilde{e}_i M$, then $\varepsilon_i(K) < \varepsilon_i(M) - 1$.

(5) For irreducible $M \in R(\nu)$-mod let $m = \varepsilon_i(M)$. Then $e_i^m M$ is isomorphic to $(\widetilde{e}_i)^m M^{\oplus [m]!}$. In particular, if $m = 1$ then $e_i M = \widetilde{e}_i M$.

(6) For irreducible modules $N \in R(\nu)$-mod and $M \in R(\nu + \alpha_i)$-mod we have $\widetilde{f}_i N \cong M$ if and only if $N \cong \widetilde{e}_i M$.

(7) Let $M, N \in R(\nu)$-mod be irreducible. Then $\widetilde{f}_i M \cong \widetilde{f}_i N$ if and only if $M \cong N$. Assuming $\varepsilon_i(M), \varepsilon_i(N) > 0$, $\widetilde{e}_i M \cong \widetilde{e}_i N$ if and only if $M \cong N$.

On the level of characters, e_i roughly removes an i from the rightmost entry of a module's character. We can construct analogous functors for removal of i from the left side of a module's character, as well as an analogue to $\widetilde{f}_i$. These are denoted by $e_i^\vee$, $\widetilde{e}_i^\vee$, $\widetilde{f}_i^\vee$ and we will use them extensively in this paper. We use the involution σ introduced below to define them. Let w_0 be the longest element of $\mathcal{S}_{|\nu|}$. Then $\sigma : R(\nu) \to R(\nu)$ is defined as follows:

$$1_{\underline{\mathbf{i}}} \mapsto 1_{w_0(\underline{\mathbf{i}})} \tag{4.47}$$

$$x_r \mapsto x_{|\nu|+1-r} \tag{4.48}$$

$$\psi_r 1_{\underline{\mathbf{i}}} \mapsto (-1)^{\delta_{i_r, i_{r+1}}} \psi_{|\nu|-r} 1_{w_0(\underline{\mathbf{i}})}. \tag{4.49}$$

For an $R(\nu)$-module M, let $\sigma^* M$ be the $R(\nu)$-module M but with the action of $R(\nu)$ twisted by σ,

$$r \cdot u = \sigma(r) u.$$

Now let $e_i^\vee : R(\nu)\text{-mod} \to R(\nu - \alpha_i)\text{-mod}$ be the restriction functor defined as

$$e_i^\vee := \sigma^* e_i \sigma = \mathrm{Res}^{R(\alpha_i)\otimes R(\nu-\alpha_i)}_{R(\nu-\alpha_i)} \circ \mathrm{Res}^{\nu}_{\alpha_i, \nu-\alpha_i}, \tag{4.50}$$

and similarly,

$$\widetilde{e}_i^\vee M := \sigma^*(\widetilde{e}_i(\sigma^* M)) = \mathrm{soc}\, e_i^\vee M, \tag{4.51}$$

$$\widetilde{f}_i^\vee M := \sigma^*(\widetilde{f}_i(\sigma^* M)) = \mathrm{cosoc}\, \mathrm{Ind}^{\nu+\alpha_i}_{\alpha_i, \nu} L(i) \boxtimes M, \tag{4.52}$$

$$\varepsilon_i^\vee(M) := \varepsilon_i(\sigma^* M) = \max\{n \geq 0 \mid (\widetilde{e}_i^\vee)^n M \neq \mathbf{0}\}. \tag{4.53}$$

Note that by the exactness of restriction, $e_i, e_i^\vee$ are exact functors, while $\widetilde{e}_i$ and $\widetilde{e}_i^\vee$ are only left exact, and $\widetilde{f}_i$ and $\widetilde{f}_i^\vee$ are only right exact. When $k \in \mathbb{N}$, the indices on e_k, $\widetilde{e}_k$, $e_k^\vee$, $\widetilde{e}_k^\vee$, $\widetilde{f}_k$, $\widetilde{f}_k^\vee$ should always be interpreted modulo ℓ, i.e. we also identify $k \in I$.

Example 4.6. The module of Example 4.4 can be constructed as $T_{0;6} = \widetilde{f}_5 \widetilde{f}_4 \widetilde{f}_3 \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$ or as $\widetilde{f}_5 \widetilde{f}_4 \widetilde{f}_3 \widetilde{f}_2 \widetilde{f}_1 L(0)$. Since $\ell = 4$ this is also $\widetilde{f}_1 \widetilde{f}_0 \widetilde{f}_3 \widetilde{f}_2 \widetilde{f}_1 \widetilde{f}_0 \mathbb{1}$, but for the purposes of this paper, we prefer the first expression.

Remark 4.7. There is a nice character-theoretic interpretation of ε_i and $\varepsilon_i^\vee$. Let M be a simple $R(\nu)$-module with $|\nu| = m$. Then

a.) $\varepsilon_i(M) = c$ implies that there exists

$$\underline{\mathbf{i}} = (i_1, \ldots, i_{m-c}, \underbrace{i, i, \ldots, i}_{c})$$

such that $1_{\underline{\mathbf{i}}} A \neq \mathbf{0}$. In other words $[\underline{\mathbf{i}}]$ is in the support of M; however no $[\underline{\mathbf{j}}]$ such that

$$\underline{\mathbf{j}} = (i_1, \dots, i_{m-c-1}, \underbrace{i, i, \dots, i}_{c+1}).$$

is in $\mathrm{supp}(M)$.

b.) $\varepsilon_i^\vee(M) = c$ implies that there exists $[\underline{\mathbf{i}}]$ in the support of M of the form

$$\underline{\mathbf{i}} = (\underbrace{i, i, \dots, i}_{c}, i_{c+1}, \dots, i_m)$$

but no $[\underline{\mathbf{j}}]$ of the form

$$\underline{\mathbf{j}} = (\underbrace{i, i, \dots, i}_{c+1}, i_{c+2}, \dots, i_m).$$

4.5. **Serre relations.** Because the functors e_i, $i \in I$, are exact, they descend to well-defined linear operators on the Grothendieck group of R, $G_0(R)$. It is shown in [16, 17] that these operators satisfy the quantum Serre relations, and that these relations are in fact minimal. We have

$$\sum_{r=0}^{-a_{ij}+1} (-1)^r e_i^{(-a_{ij}+1-r)} e_j e_i^{(r)} [M] = \mathbf{0}. \tag{4.54}$$

for all $i \neq j \in I$ and $M \in R\text{-}\mathrm{mod}$, where $e_i^{(r)} = \frac{1}{[r]!} e_i^r$ is the divided power. (Recall $a_{ij} = \langle h_i, \alpha_j \rangle$.) The minimality of these relations imply that, for $0 \leq c < -a_{ij} + 1$,

$$\sum_{r=0}^{c} (-1)^r e_i^{(c-r)} e_j e_i^{(r)} \tag{4.55}$$

is never the zero operator on $G_0(R)$ by the quantum Gabber-Kac Theorem [23] and the work of [16, 17], which essentially computes the kernel of the map from the free algebra on generators e_i to $G_0(R)$.

4.6. **Jump.** When we apply $\widetilde{f}_i$ to irreducible $R(\nu)$-module M for $i \in I$, then Proposition 4.5.2 tells us that $\widetilde{f}_i M$ is an irreducible $R(\nu + \alpha_i)$-module with

$$\varepsilon_i(\widetilde{f}_i M) = \varepsilon_i(M) + 1. \tag{4.56}$$

We could also ask whether $\varepsilon_i^\vee(\widetilde{f}_i M)$ and $\varepsilon_i^\vee(M)$ differ. Questions like this motivate the introduction of the function jump_i, which is based on a concept for Hecke algebras in [8], and was introduced for KLR algebras and studied extensively in [21].

Definition 4.8. Let M be a simple $R(\nu)$-module, and let $i \in I$. Then

$$\mathrm{jump}_i(M) := \max\{J \geq 0 \mid \varepsilon_i^\vee(M) = \varepsilon_i^\vee(\widetilde{f}_i^J M)\}. \tag{4.57}$$

Lemma 4.9. [21] *Let M be a simple $R(\nu)$-module. The following are equivalent:*

(1) $\mathit{jump}_i(M) = 0$
(2) $\widetilde{f}_i M \cong \widetilde{f}_i^\vee M$
(3) $\mathrm{Ind}\, M \boxtimes L(i^m)$ *is irreducible for all* $m \geq 1$
(4) $\mathrm{Ind}\, M \boxtimes L(i^m) = \mathrm{Ind}\, L(i^m) \boxtimes M$ *for all* $m \geq 1$
(5) $\mathit{wt}_i(M) + \varepsilon_i(M) + \varepsilon_i^\vee(M) = 0$, *where* $\mathit{wt}_i(M) = -\langle h_i, \nu \rangle$.
(6) $\varepsilon_i(\widetilde{f}_i^\vee M) = \varepsilon_i(M) + 1$
(7) $\varepsilon_i^\vee(\widetilde{f}_i M) = \varepsilon_i^\vee(M) + 1$

Proof. See [21]. □

It is shown in [21] that

$$\mathrm{jump}_i(\widetilde{f}_i M) = \max\{0, \mathrm{jump}_i(M) - 1\} = \mathrm{jump}_i(\widetilde{f}_i^\vee M). \tag{4.58}$$

It is also shown in [21] that

$$\mathrm{jump}_i(M) = \mathrm{wt}_i(M) + \varepsilon_i(M) + \varepsilon_i^\vee(M). \tag{4.59}$$

Using information from jump_i we can also determine when the crystal operators commute with their σ-symmetric versions.

Example 4.10. Suppose $\ell > 2$. Observe $\mathrm{jump}_1(L(0)) = 1$ and

$$\widetilde{f}_1^\vee \widetilde{f}_1 L(0) \cong \mathrm{Ind}\, L(1) \boxtimes T(0,1) \tag{4.60}$$

whose character has support $\{[1,0,1],[0,1,1],[0,1,1]\}$. However

$$\widetilde{f}_1 \widetilde{f}_1^\vee L(0) \cong \mathrm{Ind}\, S(1,0) \boxtimes L(1) \tag{4.61}$$

whose character has support $\{[1,0,1],[1,1,0],[1,1,0]\}$.

In the case $\ell = 2$, note $\mathrm{jump}_1(T(0,1)) = 1$ and we can similarly calculate $\widetilde{f}_1^\vee \widetilde{f}_1 T(0,1) \ncong \widetilde{f}_1 \widetilde{f}_1^\vee T(0,1)$ (in fact the former is 8-dimensional while the latter is 4-dimensional).

We shall see below that this phenomenon is special to $\mathrm{jump}_i(M) = 1$.

Lemma 4.11. *Let M be a simple $R(\nu)$-module.*

(1) [21] *If $i \neq j$, then*
   (a) $\widetilde{f}_i \widetilde{f}_j^\vee M \cong \widetilde{f}_j^\vee \widetilde{f}_i M$.
   (b) If $\widetilde{e}_j^\vee M \neq \mathbf{0}$ then $\widetilde{f}_i \widetilde{e}_j^\vee M \cong \widetilde{e}_j^\vee \widetilde{f}_i M$.
   (c) If $\widetilde{e}_j M \neq \mathbf{0}$ then $\widetilde{f}_i^\vee \widetilde{e}_j M \cong \widetilde{e}_j \widetilde{f}_i^\vee M$.
   (d) If further $\widetilde{e}_i M \neq \mathbf{0}$ then, $\widetilde{e}_i \widetilde{e}_j^\vee(M) \cong \widetilde{e}_j^\vee \widetilde{e}_i(M)$.

(2) *(a) $\mathrm{jump}_i(M) \neq 1$ if and only if $\widetilde{f}_i^\vee \widetilde{f}_i M \cong \widetilde{f}_i \widetilde{f}_i^\vee M$.*
   (b) If $\widetilde{e}_i^\vee M \neq \mathbf{0}$, then $\mathrm{jump}_i(\widetilde{e}_i^\vee M) \neq 1$ if and only if $\widetilde{e}_i^\vee \widetilde{f}_i M \cong \widetilde{f}_i \widetilde{e}_i^\vee M$.
   (c) If $\widetilde{e}_i M \neq \mathbf{0}$, then $\mathrm{jump}_i(\widetilde{e}_i M) \neq 1$ if and only if $\widetilde{e}_i \widetilde{f}_i^\vee M \cong \widetilde{f}_i^\vee \widetilde{e}_i M$.

Proof.

(1) Consider the short exact sequence,

$$\mathbf{0} \to K \to \mathrm{Ind}\, M \boxtimes L(i) \to \widetilde{f}_i M \to \mathbf{0} \tag{4.62}$$

and recall $\widetilde{f}_i M$ is the unique composition factor of $\mathrm{Ind}\, M \boxtimes L(i)$ such that $\varepsilon_i(\widetilde{f}_i M) = \varepsilon_i(M) + 1$, and that for all composition factors N of K, $\varepsilon_i(N) \leq \varepsilon_i(M)$. By the exactness of induction there is a second short exact sequence

$$\mathbf{0} \to \mathrm{Ind}\, L(j) \boxtimes K \to \mathrm{Ind}\, L(j) \boxtimes M \boxtimes L(i) \to \mathrm{Ind}\, L(j) \boxtimes \widetilde{f}_i M \to \mathbf{0}, \tag{4.63}$$

and since $i \neq j$ the Shuffle Lemma tells us that for all composition factors N' of $\mathrm{Ind}\, L(j) \boxtimes K$, $\varepsilon_i(N') \leq \varepsilon_i(M)$. By the Shuffle Lemma and Frobenius reciprocity

$$\varepsilon_i(\widetilde{f}_j^\vee \widetilde{f}_i M) = \varepsilon_i(\widetilde{f}_i \widetilde{f}_j^\vee M) = \varepsilon_i(M) + 1. \tag{4.64}$$

Hence there can be no nonzero map

$$\mathrm{Ind}\, L(j) \boxtimes K \to \widetilde{f}_i \widetilde{f}_j^\vee M, \tag{4.65}$$

so that the submodule $\mathrm{Ind}\, L(j) \boxtimes K$ is contained in the kernel of β, as pictured in (4.66).

$$\begin{tikzcd}
 & \operatorname{Ind} L(j)\boxtimes \widetilde{f}_i M \arrow[r, two heads] & \widetilde{f}_j^\vee \widetilde{f}_i M \\
\operatorname{Ind} L(j)\boxtimes M\boxtimes L(i) \arrow[ur, two heads] \arrow[dr, two heads] \arrow[urr, bend left, two heads, "\alpha"] \arrow[drr, bend right, two heads, "\beta"'] & & \\
 & \operatorname{Ind}\widetilde{f}_j^\vee M\boxtimes L(i) \arrow[r, two heads] & \widetilde{f}_i\widetilde{f}_j^\vee M
\end{tikzcd} \tag{4.66}$$

Hence β induces a nonzero map (necessarily surjective)

$$\operatorname{Ind} L(j)\boxtimes \widetilde{f}_i M \twoheadrightarrow \widetilde{f}_i\widetilde{f}_j^\vee M. \tag{4.67}$$

Because $\operatorname{Ind} L(j)\boxtimes \widetilde{f}_i M$ has unique simple quotient $\widetilde{f}_j^\vee \widetilde{f}_i M$, then $\widetilde{f}_j^\vee \widetilde{f}_i M \cong \widetilde{f}_i\widetilde{f}_j^\vee M$. This proves 1a.

The three isomorphisms in 1b, 1c, and 1d all follow from 1a. For example, if $\widetilde{e}_j^\vee M$ is nonzero, then

$$\widetilde{f}_i M \cong \widetilde{f}_i\widetilde{f}_j^\vee \widetilde{e}_j^\vee M \cong \widetilde{f}_j^\vee \widetilde{f}_i \widetilde{e}_j^\vee M. \tag{4.68}$$

Applying $\widetilde{e}_j^\vee$ to both sides we get 1b. 1c and 1d follow similarly.

(2) We prove 2a. Let $c = \varepsilon_i^\vee(M), m = \varepsilon_i(M)$.

- Suppose $\operatorname{jump}_i(M) = 0$. Then also $\operatorname{jump}_i(\widetilde{f}_i M) = \operatorname{jump}_i(\widetilde{f}_i^\vee M) = 0$ by (4.58). Thus by Lemma 4.9

$$\widetilde{f}_i^\vee \widetilde{f}_i M \cong \widetilde{f}_i \widetilde{f}_i M \cong \widetilde{f}_i \widetilde{f}_i^\vee M. \tag{4.69}$$

- Suppose $\operatorname{jump}_i(M) = 1$. By Lemma 4.9 and Proposition 4.5, $\varepsilon_i(\widetilde{f}_i M) = m+1$ but $\varepsilon_i^\vee(\widetilde{f}_i M) = c$. While $\varepsilon_i(\widetilde{f}_i^\vee M) = m$ but $\varepsilon_i^\vee(\widetilde{f}_i^\vee M) = c+1$. Further by (4.58) $\operatorname{jump}_i(\widetilde{f}_i M) = \operatorname{jump}_i(\widetilde{f}_i^\vee M) = 0$. Hence $\varepsilon_i(\widetilde{f}_i^\vee \widetilde{f}_i M) = m+2$, $\varepsilon_i^\vee(\widetilde{f}_i^\vee \widetilde{f}_i M) = c+1$ whereas $\varepsilon_i(\widetilde{f}_i \widetilde{f}_i^\vee M) = m+1$, $\varepsilon_i^\vee(\widetilde{f}_i \widetilde{f}_i^\vee M) = c+2$. Thus the two modules cannot be isomorphic.
- Suppose $\operatorname{jump}_i(M) \geq 2$. Then $\operatorname{jump}_i(\widetilde{f}_i M) = \operatorname{jump}_i(\widetilde{f}_i^\vee M) \geq 1$. We calculate

$$\varepsilon_i(\widetilde{f}_i \widetilde{f}_i^\vee M) = m+1 = \varepsilon_i(\widetilde{f}_i^\vee \widetilde{f}_i M) \tag{4.70}$$

$$\varepsilon_i^\vee(\widetilde{f}_i \widetilde{f}_i^\vee M) = c+1 = \varepsilon_i^\vee(\widetilde{f}_i^\vee \widetilde{f}_i M). \tag{4.71}$$

We will show there is no nonzero map

$$\operatorname{Ind} L(i)\boxtimes K \to \widetilde{f}_i \widetilde{f}_i^\vee M \tag{4.72}$$

for any proper submodule $K \subseteq \operatorname{Ind} M\boxtimes L(i)$. Given we have a surjection

$$\operatorname{Ind} L(i)\boxtimes M\boxtimes L(i) \twoheadrightarrow \widetilde{f}_i \widetilde{f}_i^\vee M \tag{4.73}$$

this means we must have a nonzero map

$$\operatorname{Ind} L(i)\boxtimes \widetilde{f}_i M \to \widetilde{f}_i \widetilde{f}_i^\vee M, \tag{4.74}$$

which will prove the lemma as

$$\widetilde{f}_i^\vee \widetilde{f}_i M = \operatorname{cosoc} \operatorname{Ind} L(i)\boxtimes \widetilde{f}_i M. \tag{4.75}$$

First note there is no nonzero map

$$\operatorname{Ind} L(i) \boxtimes \widetilde{f}_i^\vee M \to \widetilde{f}_i \widetilde{f}_i^\vee M \tag{4.76}$$

as $\operatorname{cosoc}(\operatorname{Ind} L(i) \boxtimes \widetilde{f}_i^\vee M) = (\widetilde{f}_i^\vee)^2 M$ and $\varepsilon_i^\vee((\widetilde{f}_i^\vee)^2 M) = c+2 \neq c+1 = \varepsilon_i^\vee(\widetilde{f}_i \widetilde{f}_i^\vee M)$. Let D be any other composition factor of $\operatorname{Ind} M \boxtimes L(i)$ apart from $\widetilde{f}_i M$ or $\widetilde{f}_i^\vee M$ (recall the latter occur with multiplicity one as composition factors). Then by Proposition 4.5, $\varepsilon_i(D) \leq m$, $\varepsilon_i^\vee(D) \leq c$. If there were a nonzero map $\operatorname{Ind} L(i) \boxtimes D \to \widetilde{f}_i \widetilde{f}_i^\vee M$, it would imply $\widetilde{f}_i^\vee D \cong \widetilde{f}_i \widetilde{f}_i^\vee M$ and so $\varepsilon_i^\vee(\widetilde{f}_i^\vee D) = c+1$ meaning $\varepsilon_i^\vee(D) = c$. Also $m+1 = \varepsilon_i(\widetilde{f}_i^\vee D) \leq \varepsilon_i(D)+1$ by the Shuffle Lemma, forcing $\varepsilon_i(D) = m$. By Lemma 4.9 this forces $0 = \operatorname{jump}_i(D)$ and $\widetilde{f}_i D \cong \widetilde{f}_i^\vee D \cong \widetilde{f}_i \widetilde{f}_i^\vee M$ from above, forcing $D \cong \widetilde{f}_i^\vee M$, which we already ruled out. Hence there must be a nonzero map

$$\operatorname{Ind} L(i) \boxtimes \widetilde{f}_i M \to \widetilde{f}_i \widetilde{f}_i^\vee M. \tag{4.77}$$

Now that we have established $\widetilde{f}_i^\vee \widetilde{f}_i M \cong \widetilde{f}_i \widetilde{f}_i^\vee M$ if and only if $\operatorname{jump}_i(M) \neq 1$, statements 2b and 2c follow directly from Proposition 4.5.6.

□

Remark 4.12. Because $\widetilde{e}_i^\vee$ and $\widetilde{f}_j$ commute for $i \neq j$, then $\varepsilon_i^\vee(\widetilde{f}_j M) = \varepsilon_i^\vee(M)$. An equivalent statement holds for $\widetilde{e}_i$, $\widetilde{f}_j^\vee$, and ε_i. When $\operatorname{jump}_i(M) \neq 0$, $\varepsilon_i^\vee(\widetilde{f}_i M) = \varepsilon_i^\vee(M)$.

5. The functor $\operatorname{pr}_\Lambda$

For $\Lambda = \sum_{i \in I} \lambda_i \Lambda_i \in P^+$ define $\mathcal{I}_\nu^\Lambda$ to be the two-sided ideal of $R(\nu)$ generated by the elements $x_1^{\lambda_{i_1}} 1_{\underline{\mathbf{i}}}$ for all $\underline{\mathbf{i}} \in \operatorname{Seq}(\nu)$. When ν is clear from the context we write, $\mathcal{I}_\nu^\Lambda = \mathcal{I}^\Lambda$. The *cyclotomic KLR algebra of weight* Λ is then defined as

$$R^\Lambda = \bigoplus_{\nu \in Q^+} R^\Lambda(\nu) \quad \text{where} \quad R^\Lambda(\nu) := R(\nu)/\mathcal{I}_\nu^\Lambda. \tag{5.1}$$

The algebra $R^\Lambda(\nu)$ is finite dimensional, [4, 21]. The category of finite dimensional $R^\Lambda(\nu)$-modules is denoted $R^\Lambda(\nu)\text{-}\mathrm{mod}$ and the category of finite dimensional R^Λ-modules is denoted $R^\Lambda\text{-}\mathrm{mod}$. The category of finite dimensional R-modules on which $\mathcal{I}^\Lambda$ vanishes is denoted

$$\operatorname{Rep}^\Lambda.$$

While we can identify $R^\Lambda\text{-}\mathrm{mod}$ with $\operatorname{Rep}^\Lambda$, we choose to work with $\operatorname{Rep}^\Lambda$. We construct a right-exact functor, $\operatorname{pr}_\Lambda : R(\nu)\text{-}\mathrm{mod} \to R(\nu)\text{-}\mathrm{mod}$, via

$$\operatorname{pr}_\Lambda M := M/\mathcal{I}^\Lambda M. \tag{5.2}$$

It is customary in the literature to interpret $\operatorname{pr}_\Lambda$ as being a functor from $R(\nu)\text{-}\mathrm{mod}$ to $R^\Lambda(\nu)\text{-}\mathrm{mod}$, but in this paper it will be more convenient to consider it as a functor $R(\nu)\text{-}\mathrm{mod} \to \operatorname{Rep}^\Lambda$. The reader may keep in mind that the image of $\operatorname{pr}_\Lambda$ consists of $R(\nu)$-modules which descend to $R^\Lambda(\nu)$-modules. Observe that in the opposite direction there is an exact functor $\operatorname{infl}_\Lambda : R^\Lambda(\nu)\text{-}\mathrm{mod} \to R(\nu)\text{-}\mathrm{mod}$, where $R(\nu)$ acts on $R^\Lambda(\nu)$-module M through the projection map $R(\nu) \twoheadrightarrow R^\Lambda(\nu)$.

Remark 5.1. If M is a $R(\nu)$-module and A is a simple module in $\operatorname{Rep}^\Lambda$ for $\Lambda \in P^+$, then since $\operatorname{pr}_\Lambda A \cong A$, the right exactness of $\operatorname{pr}_\Lambda$ implies that any surjection $M \twoheadrightarrow A$ gives a surjection $\operatorname{pr}_\Lambda M \twoheadrightarrow A$. Similarly, since there always exists a surjection $M \twoheadrightarrow \operatorname{pr}_\Lambda M$, given a surjection $\operatorname{pr}_\Lambda M \twoheadrightarrow A$ we immediately get a surjection $M \twoheadrightarrow A$. In such situations there is an equivalence between the two surjections $M \twoheadrightarrow A$ and $\operatorname{pr}_\Lambda M \twoheadrightarrow A$ which we will henceforth use freely.

If M is simple then either $\operatorname{pr}_\Lambda M = \mathbf{0}$ or $\operatorname{pr}_\Lambda M = M$. There is a useful criterion for determining the action of $\operatorname{pr}_\Lambda$ on simple $R(\nu)$-modules given by the following proposition.

Proposition 5.2. [21] Let $\Lambda = \sum_{i\in I} \lambda_i \Lambda_i \in P^+$, $\nu \in Q^+$, and let M be a simple $R(\nu)$-module. Then $\mathcal{I}^\Lambda M = \mathbf{0}$ if and only if $\operatorname{pr}_\Lambda M = M$ if and only if $\operatorname{pr}_\Lambda M \neq \mathbf{0}$ if and only if

$$\varepsilon_i^\vee(M) \leq \lambda_i$$

for all $i \in I$. When these conditions hold $M \in \mathrm{Rep}^\Lambda$. Hence we may identify M with $\operatorname{pr}_\Lambda M$ (or as an $R^\Lambda(\nu)$-module).

In this paper we will primarily consider $\Lambda = \Lambda_i$ in which case $\mathcal{I}_\nu^{\Lambda_i}$ is generated by $x_1 1_{ii_2\dots i_m}$ and $1_{ji_2\dots i_m}, j \neq i$ ranging over $\underline{\mathbf{i}} \in \mathrm{Seq}(\nu)$.

Notice that Proposition 5.2 immediately tells us that the 1-dimensional modules $T_{i;k} \in \mathrm{Rep}^{\Lambda_i}$ for any $k \geq 0$. For $\Lambda = \sum_{i\in I} \lambda_i \Lambda_i \in P^+$ and M an irreducible $R(\nu)$-module set

$$\varphi_i^\Lambda(M) = \lambda_i + \varepsilon_i(M) + \mathrm{wt}_i(M). \tag{5.3}$$

Notice that when $\Lambda = \Lambda_j$ this gives

$$\varphi_i^{\Lambda_j}(M) = \delta_{ij} + \varepsilon_i(M) + \mathrm{wt}_i(M). \tag{5.4}$$

Remark 5.3. By formula (5.4) if M is a simple module in Rep^{Λ_j} it follows that

$$\varphi_i^{\Lambda_j}(M) = \begin{cases} \delta_{ij} & \text{if } M = \mathbb{1}, \\ \mathrm{jump}_i(M) & \text{otherwise.} \end{cases} \tag{5.5}$$

Proposition 5.4. Let M be a simple $R(\nu)$-module with $\operatorname{pr}_\Lambda M \neq \mathbf{0}$. Then

$$\varphi_i^\Lambda(M) = \max\{n \in \mathbb{Z} \mid \operatorname{pr}_\Lambda \widetilde{f}_i^n M \neq \mathbf{0}\}. \tag{5.6}$$

From property (4.58) of jump_i it is clear that if we apply $\widetilde{f}_i$ sufficiently many times to any module $M \in R^\Lambda(\nu)\text{-mod}$, then eventually we will eventually reach an n for which

$$\varepsilon_i^\vee(\widetilde{f}_i^n M) > \lambda_i \tag{5.7}$$

and so $\operatorname{pr}_\Lambda \widetilde{f}_i^n M = \mathbf{0}$. Proposition 5.4 shows that φ_i^Λ measures this for simple modules in Rep^Λ. In fact it is true that $\operatorname{pr}_\Lambda M \neq \mathbf{0}$ if and only if $\varphi_i^\Lambda(M) \geq 0$ for all $i \in I$. We remark below that the function φ_i^Λ is part of a crystal datum.

5.0.1. *Module-theoretic model of* $B(\Lambda)$. Let M be a simple $R(\nu)$-module. Set

$$\mathrm{wt}(M) = -\nu \quad \text{and} \quad \mathrm{wt}_i(M) = -\langle h_i, \nu\rangle. \tag{5.8}$$

Let $\mathrm{Irr}\, R$ be the set of isomorphism classes of simple R-modules and $\mathrm{Irr}\, R^\Lambda$ be the set of isomorphism classes of simple modules in Rep^Λ. In [21] it was shown that the tuple $(\mathrm{Irr}\, R, \varepsilon_i, \varphi_i, \widetilde{e}_i, \widetilde{f}_i, \mathrm{wt})$ defines a crystal isomorphic to $B(\infty)$ and $(\mathrm{Irr}\, R^\Lambda, \varepsilon_i, \varphi_i^\Lambda, \widetilde{e}_i, \widetilde{f}_i, \mathrm{wt})$ defines a crystal isomorphic to the highest weight crystal $B(\Lambda)$.

5.1. **Interaction of** $\operatorname{pr}_\Lambda$ **and induction.** The following is a list of useful facts about the way that the functor $\operatorname{pr}_\Lambda$ interacts with the functor of induction.

Proposition 5.5. Fix $\Lambda \in P^+$, let $\mu, \nu \in Q^+$, M be a simple $R(\mu)$-module and N a simple $R(\nu)$-module.

(a) If $\operatorname{pr}_\Lambda M = \mathbf{0}$ then $\operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes N = \mathbf{0}$.
(b) If $\operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes L(i^c) = \mathbf{0}$ and $\varepsilon_i^\vee(N) \geq c$ then $\operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes N = \mathbf{0}$.
(c) If $c > \varphi_i^\Lambda(M)$ then $\operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes L(i^c) = \mathbf{0}$.
(d) Let $\varphi = \varphi_i^\Lambda(M)$, then $\operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes L(i^\varphi) \cong \widetilde{f}_i^\varphi M$.
(e) If $\operatorname{pr}_\Lambda C = M$ then $\operatorname{pr}_\Lambda \mathrm{Ind}\, C \boxtimes N \cong \operatorname{pr}_\Lambda \mathrm{Ind}\, M \boxtimes N$

Proof. We let

$$\Lambda = \sum_{i \in I} \lambda_i \Lambda_i. \tag{5.9}$$

(a) If $\mathrm{pr}_\Lambda M = \mathbf{0}$, then by Proposition 5.2 there is some $i \in I$ such that $\varepsilon_i^\vee(M) > \lambda_i$. Suppose that $\mathrm{pr}_\Lambda \operatorname{Ind} M \boxtimes N \neq \mathbf{0}$. Then it has some simple quotient Q, and there are surjections

$$\operatorname{Ind} M \boxtimes N \twoheadrightarrow \mathrm{pr}_\Lambda \operatorname{Ind} M \boxtimes N \twoheadrightarrow Q. \tag{5.10}$$

Frobenius reciprocity and Proposition 5.2 imply that $\mathrm{pr}_\Lambda Q = Q$. By Frobenius reciprocity $\operatorname{Res}^{\mu+\nu}_{\mu,\nu} Q$ has $M \boxtimes N$ as a $(R(\mu) \otimes R(\nu))$-submodule. But Remark 4.7 then implies $\varepsilon_i^\vee(Q) > \lambda_i$ so that $\mathrm{pr}_\Lambda Q = \mathbf{0}$, a contradiction.

(b) If $\varepsilon_i^\vee(N) \geq c$ then there is a surjection,

$$\operatorname{Ind} L(i^c) \boxtimes (\widetilde{e}_i^\vee)^c N \twoheadrightarrow N \tag{5.11}$$

and by the exactness of induction a surjection

$$\operatorname{Ind} M \boxtimes L(i^c) \boxtimes (\widetilde{e}_i^\vee)^c N \twoheadrightarrow \operatorname{Ind} M \boxtimes N. \tag{5.12}$$

If $\mathrm{pr}_\Lambda \operatorname{Ind} M \boxtimes L(i^c) = \mathbf{0}$, then by part (a) above and the right exactness of pr_Λ, $\mathrm{pr}_\Lambda \operatorname{Ind} M \boxtimes N = \mathbf{0}$.

(c) This follows from Proposition 5.4 and the fact that the induced module has unique simple quotient $\widetilde{f}_i^c M$; or see [21].

(d) Consider the exact sequence,

$$\mathbf{0} \to K \to \operatorname{Ind} M \boxtimes L(i^\varphi) \to \widetilde{f}_i^\varphi M \to \mathbf{0}. \tag{5.13}$$

$\widetilde{f}_i^\varphi M$ is the unique composition factor of $\operatorname{Ind} M \boxtimes L(i^\varphi)$ such that $\varepsilon_i(\widetilde{f}_i^\varphi M) = \varphi + \varepsilon_i(M)$, so $\varepsilon_i(D) < \varphi + \varepsilon_i(M)$ for all composition factors D of K by Proposition 4.5. All composition factors D of K have the same weight as $\widetilde{f}_i^\varphi M$. By (5.3) and Proposition 4.5, $\varphi_i^\Lambda(D) = \lambda_i + \varepsilon_i(D) + \mathrm{wt}_i(D) < \lambda_i + \varepsilon_i(\widetilde{f}_i^\varphi M) + \mathrm{wt}_i(\widetilde{f}_i^\varphi M) = \varphi_i^\Lambda(\widetilde{f}_i^\varphi M) = 0$. In particular this shows $\mathrm{pr}_\Lambda K = \mathbf{0}$ so by the right exactness of pr_Λ we get (d).

(e) Consider the diagram in (5.14),

$$\begin{tikzcd}
 & & \mathbf{0} \arrow[d] & & \\
 & & \operatorname{Ind} \mathcal{I}^\Lambda C \boxtimes N \arrow[d, "\alpha"] \arrow[dr, "\beta \circ \alpha"] & & \\
\mathbf{0} \arrow[r] & \mathcal{I}^\Lambda(\operatorname{Ind} C \boxtimes N) \arrow[r] & \operatorname{Ind} C \boxtimes N \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathrm{pr}_\Lambda(\operatorname{Ind} C \boxtimes N) \arrow[r] & \mathbf{0} \\
 & & \operatorname{Ind} M \boxtimes N \arrow[ur, dotted, "g"'] \arrow[d] & & \\
 & & \mathbf{0} & &
\end{tikzcd} \tag{5.14}$$

where the horizontal and vertical sequences are exact. Recall that $\mathcal{I}^{\Lambda}_{\mu}$ in $R(\mu)$ is generated by the set $\{x_1^{\lambda_{i_1}} 1_{\underline{\mathbf{i}}}\}_{\underline{\mathbf{i}}\in \mathrm{Seq}(\mu)}$ where $\underline{\mathbf{i}} = i_1 i_2 \dots i_m$ and $|\mu| = m$. Under the embedding

$$R(\mu) \hookrightarrow R(\mu) \otimes R(\nu) \hookrightarrow R(\mu+\nu), \tag{5.15}$$

this set maps to the set

$$\Big\{ \sum_{\underline{\mathbf{j}}\in \mathrm{Seq}(\nu)} x_1^{\lambda_{i_1}} 1_{\underline{\mathbf{ij}}} \Big\}_{\underline{\mathbf{i}}\in \mathrm{Seq}(\mu)} \tag{5.16}$$

in $R(\mu+\nu)$. This set is contained in the ideal generated by $\{x_1^{\lambda_{i_1}} 1_{\underline{\mathbf{k}}}\}_{\underline{\mathbf{k}}\in \mathrm{Seq}(\mu+\nu)}$ which generates $\mathcal{I}^{\Lambda}_{\mu+\nu}$. It follows that

$$R(\mu+\nu)\mathcal{I}^{\Lambda}_{\mu} \subseteq \mathcal{I}^{\Lambda}_{\mu+\nu}, \tag{5.17}$$

and hence

$$\mathrm{Ind}\, \mathcal{I}^{\Lambda}_{\mu} C \boxtimes N \subseteq \mathcal{I}^{\Lambda}_{\mu+\nu} (\mathrm{Ind}\, C \boxtimes N). \tag{5.18}$$

This tells us that the composition $\beta \circ \alpha$ from the diagram in (5.14) is zero, so there exists a surjective homomorphism $g : \mathrm{Ind}\, M \boxtimes N \to \mathrm{pr}_{\Lambda} \mathrm{Ind}\, C \boxtimes N$. Applying pr_{Λ} to the diagram (5.14), and denoting the resulting maps from γ, β, and g as $\tilde{\gamma}$, $\tilde{\beta}$, and $\tilde{g}$ respectively, right exactness yields $\tilde{\gamma}$, $\tilde{\beta}$, and $\tilde{g}$ are surjections as shown in (5.19). It follows from considerations of dimension and that $\mathrm{pr}_{\Lambda} C = M$ that $\tilde{g}$ must be an isomorphism.

$$\begin{tikzcd}
\mathrm{pr}_{\Lambda}(\mathrm{Ind}\, C \boxtimes N) \arrow[r, "\tilde{\beta}"] \arrow[d, "\tilde{\gamma}"'] & \mathrm{pr}_{\Lambda}(\mathrm{Ind}\, C \boxtimes N) \arrow[r] & \mathbf{0} \\
\mathrm{pr}_{\Lambda}(\mathrm{Ind}\, M \boxtimes N) \arrow[ur, "\tilde{g}"'] \arrow[d] & & \\
\mathbf{0} & &
\end{tikzcd} \tag{5.19}$$

□

5.2. **Applying Proposition 5.5 to $T_{i;k}$.** We will frequently need to compute jump_j for the 1-dimensional "trivial" $R(\gamma^{+}_{i;k})$-module $T_{i;k}$. When $k = 0$, we compute for the unit module that $\mathrm{jump}_j(\mathbb{1}) = 0$ but $\varphi_j^{\Lambda_i}(\mathbb{1}) = \delta_{ij}$. When $k \geq 1$,

$$\mathrm{wt}_j(T_{i;k}) = -\langle h_j, \gamma^{+}_{k;i}\rangle = \delta_{j,i-1} - \delta_{j,i} + \delta_{j,i+k} - \delta_{j,i+k-1}. \tag{5.20}$$

Note that here as elsewhere, the indices p, q in $\delta_{p,q}$ should be taken modulo ℓ. Then,

$$\mathrm{jump}_j(T(i, i+1, \dots, i+k-1)) = \delta_{j,i-1} + \delta_{j,i+k}. \tag{5.21}$$

Similarly,

$$\varphi_j^{\Lambda_i}(T(i, i+1, \dots, i+k-1)) = \delta_{j,i-1} + \delta_{j,i+k}. \tag{5.22}$$

Here we record some useful facts concerning the way that the modules $T_{0;k}$ interact with the functors induction and pr_{Λ_0}. Notice that all these facts hold for $T_{i;k}$ and pr_{Λ_i} after making obvious modifications.

Proposition 5.6. Fix $k \in \mathbb{N}$, $k > 0$.

1. If $j \not\equiv -1, k$ then $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T(0, 1, \dots, k-1) \boxtimes L(j) = \mathbf{0}$.
2. If $k \not\equiv -1$ then $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T(0, 1, \dots, k-1) \boxtimes L(k) \cong T(0, 1, \dots, k)$.
3. If $k \not\equiv -1$ then $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T(0, 1, \dots, k-1) \boxtimes L(k) \boxtimes L(k) = \mathbf{0}$.

4. $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \cong \Big(\operatorname{Ind} T_{0;k} \boxtimes L(-1)\Big) \Big/ T(-1,0,\dots,k-1)$ and further
   (a) If $k \not\equiv -1$ then $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1)$ is irreducible and
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes L(-1) = \mathbf{0}.$$
   (b) If $k \equiv -1$, then
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes L(-1) \cong \widetilde{f}_{-1}^{2} T_{0;k}$$
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes L(-1) \boxtimes L(-1) = \mathbf{0}.$$
   Further, if $k > 1$ then $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1)$ has two composition factors. But if $k = 1$ (so $\ell = 2$) then $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) = T(0,1)$ is irreducible.

Proof. Note the hypothesis $k > 0$ implies $T_{0;k} \neq \mathbb{1}$, i.e. $\nu \neq 0$.
Below we write $=$ for $\equiv \bmod \ell$ or equality in I.

1. When $j \neq -1, k$, formula (5.22) gives
$$\varphi_j^{\Lambda_0}(T(0,1,\dots,k-1)) = 0. \tag{5.23}$$
Hence by Proposition 5.5.c
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T(0,1,\dots,k-1) \boxtimes L(j) = \mathbf{0}. \tag{5.24}$$
2. When $k \neq -1$, formula (5.22) gives
$$\varphi_k^{\Lambda_0}(T(0,1,\dots,k-1)) = 1. \tag{5.25}$$
Hence by Proposition 5.5.d
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T(0,1,\dots,k-1) \boxtimes L(k) \cong \widetilde{f}_k T(0,1,\dots,k-1) \tag{5.26}$$
$$\cong T(0,1,\dots,k-1,k), \tag{5.27}$$
where the second isomorphism holds by Frobenius reciprocity and the irreducibility of $\widetilde{f}_k T_{0;k}$.
3. As noted above
$$\varphi_k^{\Lambda_0}(T(0,\dots,k-1)) = 1. \tag{5.28}$$
Proposition 5.5.c then implies,
$$\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T(0,\dots,k-1) \boxtimes L(k) \boxtimes L(k) = \mathbf{0}. \tag{5.29}$$
4. Let $v \otimes u$ span the 1-dimensional module $T_{0;k} \boxtimes L(-1)$. Then as in (4.35), $M := \operatorname{Ind} T_{0;k} \boxtimes L(-1)$ has basis
$$\{\ 1_{\underline{\mathbf{i}}} \otimes (v \otimes u),\ \psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u),\ \dots,\ \psi_1 \cdots \psi_{k-1}\psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u)\ \} \tag{5.30}$$
where $\underline{\mathbf{i}} = (0,1,\dots,k-1,-1)$.
   Recall that $\mathcal{I}^{\Lambda_0}$ is generated by $x_1 1_{\underline{\mathbf{j}}}$ where $j_1 = 0$ and by $1_{\underline{\mathbf{p}}}$ where $p_1 \neq 0$.
   In particular, for $\underline{\mathbf{p}} = (-1,0,\dots,k-1)$ we see
$$1_{\underline{\mathbf{p}}}\Big(\psi_1 \dots \psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u)\Big) = \psi_1 \dots \psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u) \in \mathcal{I}^{\Lambda_0} M, \tag{5.31}$$
but $1_{\underline{\mathbf{p}}}(\psi_r \dots \psi_{k-1}\psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u)) = 0$ for $r > 1$. Also note $1_{\underline{\mathbf{j}}} 1_{\underline{\mathbf{p}}} = 0$. We further calculate
$$x_1 1_{\underline{\mathbf{j}}}\Big(\psi_r \dots \psi_r 1_{\underline{\mathbf{i}}} \otimes (v \otimes u)\Big) = 1_{\underline{\mathbf{j}}} \psi_r \dots \psi_k 1_{\underline{\mathbf{i}}} \otimes (x_1 v \otimes u) = 0 \tag{5.32}$$
whenever $r > 1$. Hence $\mathcal{I}^{\Lambda_0} M$ is spanned by $\psi_1 \dots \psi_k 1_{\underline{\mathbf{i}}} \otimes (v \otimes u)$ and so $\mathrm{pr}_{\Lambda_0} M = M \big/ T(-1,0,\dots,k-1)$ as stated.
   4(a) Suppose $k \neq -1$. As $\varphi_{-1}^{\Lambda_0}(T_{0;k}) = 1$ by (5.22), Proposition 5.5.d tells us that $\mathrm{pr}_{\Lambda_0} M = \widetilde{f}_{-1} T_{0;k}$ is irreducible. In particular it has dimension k. Using the Shuffle Lemma (along with the calculation of $\mathcal{I}^{\Lambda_0} M$ above), one could easily compute its character. By Proposition 5.5.c we see $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes L(-1) = \mathbf{0}$.

4(b) Next suppose $k = -1$ (i.e. $k \equiv -1 \bmod \ell$) and $k > 1$ (when considering $k \in \mathbb{N}$). Then (5.22) yields $\varphi_{-1}^{\Lambda_0}(T_{0;k}) = 2$ so Proposition 5.5.d immediately gives (4b). Further, it is easy to see $\widetilde{f}_{-1}T_{0;k} = \widetilde{f}_k T_{0;k} = T_{0;k+1}$ which is 1-dimensional so from above $\mathrm{pr}_{\Lambda_0} M$ has at least 2 composition factors when $k > 1$. Next, the Shuffle Lemma and Serre relations (4.55) tell us the $(k-1)$-dimensional subquotient, corresponding to the span of

$$\{\psi_k \otimes (v \otimes u), \ldots, \psi_2 \cdots \psi_k \otimes (v \otimes u)\}, \tag{5.33}$$

is irreducible.

The remaining case is $k \equiv -1 \bmod \ell$, $k = 1$, forcing $\ell = 2$, and also $T_{0;k} = L(0)$. As above $\varphi_{-1}^{\Lambda_0}(T_{0;1}) = 2$, yielding (4b). The only difference is that $\mathrm{pr}_{\Lambda_0} M$ has only 1 composition factor (namely $T_{0;k+1}$) as it is only 1-dimensional.

□

6. Main theorems

As remarked in Section 5.0.1, the graph with nodes corresponding to isomorphism classes $[M]$ for M a simple $R^{\Lambda_i}(\nu)$-module and arrows $[\widetilde{e}_j M] \xrightarrow{j} [M]$ is the crystal graph $B(\Lambda_i)$. We can also use ℓ-restricted partitions λ to label the nodes of $B(\Lambda_i)$ as $[M^\lambda]$. The main theorems show for the isomorphism $\mathcal{T} : B(\Lambda_i) \xrightarrow{\simeq} \mathcal{B} \otimes B(\Lambda_{i-1})$ that $k-1+i \otimes \mu = \mathcal{T}(\lambda)$ corresponds to

$$\mathrm{Ind}\, T_{i;k} \boxtimes [M^\mu] \twoheadrightarrow [M^\lambda]$$

for $k = r(M^\lambda)$ (defined below), and that the crystal operators commute with this surjection in the appropriate manner.

Another way to view the theorems is that they give a module-theoretic construction of $\mathcal{T}$ and justify it is an isomorphism of crystals.

Theorem 6.1. *Let A be a simple $R(\nu)$-module in Rep^{Λ_i} with $|\nu| \geq 1$.*

(1) There exists $k \in \mathbb{N}, k \geq 1$ such that $\widetilde{e}_{i+k-1}^\vee \ldots \widetilde{e}_{i+1}^\vee \widetilde{e}_i^\vee A$ is a simple $R(\nu - \gamma_{i;k}^+)$-module in $\mathrm{Rep}^{\Lambda_{i-1}}$.

(2) Let

$$r(A) = k$$

be the minimal k such that statement (1) holds and let

$$\mathcal{R}(A) = \widetilde{e}_{i+k-1}^\vee \ldots \widetilde{e}_{i+1}^\vee \widetilde{e}_i^\vee A.$$

Then there exists a surjection

$$\mathrm{pr}_{\Lambda_i} \mathrm{Ind}\, T(i, i+1, \ldots, i+k-1) \boxtimes \mathcal{R}(A) \twoheadrightarrow A. \tag{6.1}$$

Proof. For ease of exposition, we set $i = 0$ in the proof. For $t \in \mathbb{N}$ set $\mathcal{R}_0(A) = A$ and let

$$\mathcal{R}_t(A) = \widetilde{e}_{t-1}^\vee \ldots \widetilde{e}_1^\vee \widetilde{e}_0^\vee A. \tag{6.2}$$

We show by induction on $t \leq r(A)$ that $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$ and there exists a surjection

$$\mathrm{Ind}\, T(0, 1, \ldots, t-1) \boxtimes \mathcal{R}_t(A) \twoheadrightarrow A. \tag{6.3}$$

In the base case $t = 0$, $\mathcal{R}_0(A) = A$. If $|\nu| = 1$, then $A = L(0)$ and $\widetilde{e}_0^\vee L(0) \cong \mathbb{1} \in \mathrm{Rep}^{\Lambda_{-1}}$, so $r(A) = 1$. The existence of the surjection in this case is vacuous. Assume that $|\nu| > 1$. Then $\mathcal{R}_1(A) = \widetilde{e}_0^\vee A \neq \mathbf{0}, \mathbb{1}$. By Proposition 4.5.2 there is a surjection

$$\mathrm{Ind}\, L(0) \boxtimes \mathcal{R}_1(A) \twoheadrightarrow A. \tag{6.4}$$

It follows directly from Proposition 5.6 and Proposition 5.5.b that if $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T_{0;t} \boxtimes D \twoheadrightarrow A$ and $t \geq 1$ then $D \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$.

In more detail, Proposition 5.2 implies $D \in \mathrm{Rep}^{\Lambda}$ where $\Lambda = \sum \varepsilon_i^\vee(D)\Lambda_i \in P^+$. Proposition 5.5.b tells us $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T_{0;t} \boxtimes D \neq 0$ implies $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T_{0;t} \boxtimes L(i^{\varepsilon_i^\vee(D)}) \neq 0$. Thus for $i \neq -1, t$ we have $\varepsilon_i^\vee(D) = 0$ by Proposition 5.6.1.

If $t \neq -1$, Proposition 5.6.4a implies $\varepsilon_{-1}^\vee(D) \leq 1$ and Proposition 5.6.3 implies $\varepsilon_t^\vee(D) \leq 1$ so $D \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$. If $t = -1$ then Proposition 5.6.4b implies $\varepsilon_{-1}^\vee(D) \leq 2$ so $D \in \mathrm{Rep}^{2\Lambda_{-1}} = \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$.

Since $\mathrm{pr}_{\Lambda_0} A = A$, observe any surjection $M \twoheadrightarrow A$ factors through $M \twoheadrightarrow \mathrm{pr}_{\Lambda_0} M \twoheadrightarrow A$. Assume our inductive hypothesis (6.3) holds. Then from above, $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$. If in fact $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_{-1}}$ then we are done (and $t \geq r(A)$). If not, then $\mathcal{R}_{t+1}(A) = \widetilde{e}_t^\vee \mathcal{R}_t(A) \neq 0$.

Transitivity and exactness of induction give us a surjection

$$\mathrm{Ind}\, T(0, \ldots, t-1) \boxtimes L(t) \boxtimes \mathcal{R}_{t+1}(A) \twoheadrightarrow A. \tag{6.5}$$

In the first case, suppose $t \neq -1$. Then Proposition 5.5.e and Proposition 5.6.2 imply

$$\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T_{0;t} \boxtimes L(t) \boxtimes \mathcal{R}_{t+1}(A) \cong \mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T_{0;t+1} \boxtimes \mathcal{R}_{t+1}(A) \tag{6.6}$$

and we get

$$\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, T(0, \ldots, t-1, t) \boxtimes \mathcal{R}_{t+1}(A) \twoheadrightarrow A. \tag{6.7}$$

In the case $t = -1$ then by the inductive hypothesis $\mathcal{R}_t(A) \in \mathrm{Rep}^{2\Lambda_{-1}}$ and $\mathcal{R}_t(A) \notin \mathrm{Rep}^{\Lambda_{-1}}$ as we are assuming $t < r(A)$. Thus $\varepsilon_{-1}^\vee(\widetilde{e}_{-1}^\vee \mathcal{R}_t(A)) = \varepsilon_{-1}^\vee(\mathcal{R}_{t+1}(A)) = 1$. If K is any composition factor of $\mathrm{Ind}\, T_{0;t} \boxtimes L(-1)$ other than $T_{0;t+1}$ then $\varphi_{-1}^{\Lambda_0}(K) \leq 0$ by (5.3) so $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, K \boxtimes L(-1) = \mathbf{0}$ which implies $\mathrm{pr}_{\Lambda_0} \mathrm{Ind}\, K \boxtimes \mathcal{R}_{t+1}(A) = \mathbf{0}$. So (6.5) must factor through

$$\mathrm{Ind}\, T(0, \ldots, -2, -1) \boxtimes \mathcal{R}_{t+1}(A) \twoheadrightarrow A. \tag{6.8}$$

This completes the induction.

We take $r(A)$ to be the smallest k such that $\mathcal{R}_k(A) \in \mathrm{Rep}^{\Lambda_{-1}}$. Note that the process above must terminate as $r(A) \leq |\nu|$. In fact, in the case $r(A) = |\nu|$ we must have $A = T_{0;|\nu|}$ and $\mathcal{R}_{|\nu|}(A) = \mathcal{R}(A) = \mathbb{1} \in \mathrm{Rep}^{\Lambda_{-1}}$. □

By considering sign in place of trivial modules, a very similar proof yields the following theorem.

Theorem 6.2. *Let A be a simple $R(\nu)$-module in Rep^{Λ_i} with $|\nu| \geq 1$.*

(1) There exists $k \in \mathbb{N}, k \geq 1$ such that $\mathbf{0} \neq \widetilde{e}_{i-k+1}^\vee \ldots \widetilde{e}_{i-1}^\vee \widetilde{e}_i^\vee A$ is a simple $R(\nu - \gamma_{i;k}^-)$-module in $\mathrm{Rep}^{\Lambda_{i+1}}$.

(2) Let $c(A) = k$ be the minimal k such that (1) holds and let $\mathcal{C}(A) = \widetilde{e}_{i-k+1}^\vee \ldots \widetilde{e}_{i-1}^\vee \widetilde{e}_i^\vee A$. Then there exists a surjection

$$\mathrm{pr}_{\Lambda_i} \mathrm{Ind}\, S(i, i-1, \ldots, i-k+1) \boxtimes \mathcal{C}(A) \twoheadrightarrow A. \tag{6.9}$$

Conjecture 6.3. With hypotheses as above,

$$A = \mathrm{cosoc}\, \mathrm{pr}_{\Lambda_i} \mathrm{Ind}\, T_{i;r(A)} \boxtimes \mathcal{R}(A),$$
$$A = \mathrm{cosoc}\, \mathrm{pr}_{\Lambda_i} \mathrm{Ind}\, S_{i;c(A)} \boxtimes \mathcal{C}(A).$$

6.1. **Relation to Specht modules.** A Specht module for $\mathcal{S}_n$ is constructed as a submodule of the induction of a trivial module from a Young subgroup $\mathcal{S}_\lambda$ (this is one of our $\mathcal{S}_P$ as in (4.34)). Specht modules can also be constructed for the Hecke algebra of type A as in [7]. They are equipped with an integral form that allows one to specialize the Specht modules over $\mathbb{F}_\ell$ in the former case, to an ℓ-th root of unity in the latter.

When λ is ℓ-regular, the specialization of the Specht module $\bar{S}^\lambda$ has unique simple quotient D^λ. In other words, D^λ is a subquotient of a module induced from a 1-dimensional module. Further $\{D^\lambda \mid \lambda \vdash n, \lambda \text{ is } \ell\text{-regular}\}$ is a complete set of simple modules of $\mathbb{F}_\ell \mathcal{S}_n$ or the finite Hecke algebra at an ℓ-th root of unity. The crystal structure on these simples by taking socle of restriction agrees

with the model of $B(\Lambda_0)$ taking nodes to be ℓ-regular partitions [18]. This is the model compatible with tensoring by $\mathcal{B}^{\mathrm{opp}}$. (See Section 3.)

Repeating the construction of Theorem 6.2 yields D^λ as the quotient of a module induced from a (possibly conjugate) parabolic subalgebra of shape λ^T, where the module being induced is a (parabolic) sign module. In other words, the restriction of D^λ to that parabolic subalgebra contains a $\boxtimes$ of sign modules $S_{i;k}$. On the other hand, in the construction of Specht modules for the finite Hecke algebra of type A given in [7], the Specht module contains a special vector that is anti-symmetrized according to a parabolic subalgebra of shape λ^T. In other words, the same induced module that has D^λ as a quotient also has a nonzero map to S^λ.

In fact $\mathbb{Q} \otimes_{\mathbb{Z}} S^\lambda$ can be characterized as the unique irreducible $\mathbb{Q}\mathcal{S}_n$-module such that $\mathrm{Res}_{\mathcal{S}_\lambda}$ contains a trivial module and $\mathrm{Res}_{\mathcal{S}_{\lambda^T}}$ contains a sign module.

When ℓ is a root of unity (or we work over $\mathbb{F}_\ell$) the difficulty is in specializing quotients of (induced) modules. The existence of a map from and induced sign module to D^λ is not a surprise, but the result on how the crystal operators act is nontrivial.

6.2. **The action of crystal operators $\widetilde{f}_j$ and $\widetilde{e}_j$.** Next we study the action of the crystal operators $\widetilde{e}_j$ and $\widetilde{f}_j$ to show (6.1) categorifies our crystal isomorphism $\mathcal{T}$. We refer the reader back to Section 3.

Compare the theorems below with (2.8) and (2.9). As in [21] simple modules correspond to nodes in $B(\Lambda_i)$. Each node of the perfect crystal $\mathcal{B}$ (respectively $\mathcal{B}^{\mathrm{opp}}$) corresponds to a family of trivial (respectively sign) modules $T_{i;k+t\ell}, t \in \mathbb{N}$. (However this does not give a categorification of $\mathcal{B}$ itself.) It is in this manner that the main theorems of this paper give a categorification of the crystal isomorphism $\mathcal{T}$ (resp. $\mathcal{T}^{\mathrm{opp}}$).

Theorem 6.4. *Let $A \in \mathrm{Rep}^{\Lambda_i}$ be simple. Let $j \in I$ be such that $\widetilde{e}_j A \neq 0$, and let $k = r(A)$. Then there exists a surjection*

$$\begin{aligned} &\mathrm{Ind}\left(\widetilde{e}_j T_{i;k} \boxtimes \mathcal{R}(A)\right) \twoheadrightarrow \widetilde{e}_j A \qquad \textit{if} \quad \varepsilon_j(T_{i;k}) > \varphi_j^{\Lambda_{i-1}}(\mathcal{R}(A)) \\ &\mathrm{Ind}\left(T_{i;k} \boxtimes \widetilde{e}_j \mathcal{R}(A)\right) \twoheadrightarrow \widetilde{e}_j A \qquad \textit{if} \quad \varepsilon_j(T_{i;k}) \leq \varphi_j^{\Lambda_{i-1}}(\mathcal{R}(A)). \end{aligned} \tag{6.10}$$

Theorem 6.5. *Let $A \in \mathrm{Rep}^{\Lambda_i}$ be simple. Let $j \in I$ be such that $\mathrm{pr}_{\Lambda_i} \widetilde{f}_j A \neq 0$, and let $k = r(A)$. Then there exists a surjection*

$$\begin{aligned} &\mathrm{Ind}\left(\widetilde{f}_j T_{i;k} \boxtimes \mathcal{R}(A)\right) \twoheadrightarrow \widetilde{f}_j A \qquad \textit{if} \quad \varepsilon_j(T_{i;k}) \geq \varphi_j^{\Lambda_{i-1}}(\mathcal{R}(A)) \\ &\mathrm{Ind}\left(T_{i;k} \boxtimes \widetilde{f}_j \mathcal{R}(A)\right) \twoheadrightarrow \widetilde{f}_j A \qquad \textit{if} \quad \varepsilon_j(T_{i;k}) < \varphi_j^{\Lambda_{i-1}}(\mathcal{R}(A)). \end{aligned} \tag{6.11}$$

Theorem 6.4 follows directly from Theorem 6.5, therefore will only prove the latter. Similar theorems hold using sign modules and $\mathcal{C}(A)$.

Before doing this, we need to establish several lemmas.

Proposition 6.6. [27] Let $m = |\nu|$. Let M be a simple $R(\nu)$-module. If $M \in \mathrm{Rep}^{\Lambda_i}$ and $\mathbf{0} \neq \widetilde{e}_i^\vee(M) \in \mathrm{Rep}^{\Lambda_{i+1}}$ then $M = T_{i;m}$.

Proof. This can be directly adapted from Theorem 3.7 of [27], replacing $\widetilde{e}_i$ with $\widetilde{e}_i^\vee$ and noting $\varepsilon_j^\vee(M) = \delta_{i,j}$, $\varepsilon_j^\vee(\widetilde{e}_i^\vee M) = \delta_{i+1,j}$ (assuming $m \geq 2$). It was proved in the context of $B(\Lambda_i)$, hence holds for Rep^{Λ_i} by [21]. □

Proposition 6.7. Let A, $\mathcal{R}_t(A)$ be as in (6.2) and $m = |\nu|$. If there exists $1 \leq t < r(A)$ with $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_{i+t}}$, then in fact $\mathcal{R}_j(A) \in \mathrm{Rep}^{\Lambda_{i+j}}$ for all $1 \leq j \leq r(A)$, $r(A) = \min\{\ell - 1, m\}$, and $A = T_{i;m}$.

Proof. As usual, we set $i = 0$ for ease of exposition. We have already shown $\mathcal{R}_t(A) \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}}$. Given $A = \mathcal{R}_0(A) \in \mathrm{Rep}^{\Lambda_0}$, suppose $e_0^\vee A = \widetilde{e}_0^\vee A = \mathcal{R}_1(A) \in \mathrm{Rep}^{\Lambda_1}$ then by Proposition 6.6,

$A = T_{0;m}$ and we are done. Further in the case $\ell = 2$ this means $r(A) = 1$. If $\ell > 2$ then $r(A) \leq \ell - 1$. Assume otherwise.

Case 1: $\ell \neq 2$. Then $\varepsilon_{-1}^{\vee}(\mathcal{R}_1(A)) = 1$. This means there is $[\underline{\mathbf{i}}] = [i_1, i_2, \dots, i_m] \in \operatorname{supp}(A)$ with $i_1 = 0$, $i_2 = -1$. Recall we have

$$\operatorname{Ind} T_{0;t-1} \boxtimes \mathcal{R}_t(A) \twoheadrightarrow A. \tag{6.12}$$

By the Shuffle Lemma, the only way to have $[\underline{\mathbf{i}}] \in \operatorname{supp}(A)$ is if $\varepsilon_{-1}^{\vee}(\mathcal{R}_t(A)) \geq 1$. *Case 2*: $\ell = 2$. Then $\varepsilon_{-1}^{\vee}(\mathcal{R}_1(A)) = 2$, as we assumed $r(A) > 1$. Then there is $[\underline{\mathbf{i}}] \in \operatorname{supp}(A)$ with $i_1 = 0$, $i_2 = -1$, $i_3 = -1$. Again, by the Shuffle Lemma, this is only possible if $\varepsilon_{-1}^{\vee}(\mathcal{R}_t(A)) \geq 1$.

Furthermore, when $t \equiv -1 \bmod \ell$ for $0 < t < r(A)$ we have $\varepsilon_{-1}^{\vee}(\mathcal{R}_t(A)) = 2$ by the minimality of $r(A)$. □

Lemma 6.8. *Let A be a simple $R^{\Lambda_0}(\nu)$-module with $k = r(A)$, $\mathbb{1} \neq \mathcal{R}(A) \in \operatorname{Rep}^{\Lambda_{-1}}$. Fix $j \in I$. Let $J = jump_j(\mathcal{R}(A))$. If $k \neq j+1$ then for $t \in \mathbb{N}$, $0 \leq t \leq k$,*

$$jump_j(\mathcal{R}_t(A)) = \begin{cases} J & t \not\equiv j+1 \bmod \ell \\ J+1 & t \equiv j+1 \bmod \ell. \end{cases} \tag{6.13}$$

If $k = j+1$ and $J \neq 0$, then for $0 \leq t \leq k$,

$$jump_j(\mathcal{R}_t(A)) = \begin{cases} J-1 & t \not\equiv j+1 \bmod \ell \\ J & t \equiv j+1 \bmod \ell. \end{cases} \tag{6.14}$$

If $k = j+1$ and $J = 0$, then for $0 < t < k$

$$jump_j(\mathcal{R}_t(A)) = \begin{cases} 0 & t \not\equiv j+1 \bmod \ell \\ 1 & t \equiv j+1 \bmod \ell. \end{cases} \tag{6.15}$$

and $jump_j(A) = 0$.

Proof. We will first prove the lemma in the case $A = T_{0;m}$, where $m = |\nu|$. Then $k \leq \ell - 1$. For $0 \leq t \leq k$ we have $\mathcal{R}_t(A) = T(t, t+1, \dots, m-1)$. From (5.21), $\operatorname{jump}_j(\mathcal{R}_t(A)) = \delta_{j,t-1} + \delta_{j,m}$ (recalling none of these modules are $\mathbb{1}$ by hypothesis). One can easily check the Lemma holds.

From now on, we assume A is not a trivial module.

We now continue with the third case. Suppose $k = j+1$ and $J = 0$. Then $\mathcal{R}(A) = \mathcal{R}_k(A) \in \operatorname{Rep}^{\Lambda_{-1}}$, so $\varepsilon_j^{\vee}(\mathcal{R}(A)) = \delta_{j,-1}$. $\mathcal{R}_{k-1}(A) = \widetilde{f}_j^{\vee}\mathcal{R}(A) = \widetilde{f}_j\mathcal{R}(A)$ so $\varepsilon_j^{\vee}(\mathcal{R}_{k-1}(A)) = \varepsilon_j^{\vee}(\mathcal{R}(A)) + 1$ and $\varepsilon_j(\mathcal{R}_{k-1}(A)) = \varepsilon_j(\mathcal{R}(A)) + 1$. In particular $\mathcal{R}_t(A) \in \operatorname{Rep}^{\Lambda_t + \Lambda_{-1}}$ but we may assume $\mathcal{R}_t(A) \notin \operatorname{Rep}^{\Lambda_t}$ or else by [27] this would force $\mathcal{R}(A)$ and A itself to be trivial. Further $\operatorname{wt}_j(\mathcal{R}_{k-1}(A)) = \operatorname{wt}_j(\widetilde{f}_j^{\vee}\mathcal{R}(A)) = \operatorname{wt}_j(\mathcal{R}(A)) - 2$. Hence

$$\begin{aligned} \operatorname{jump}_j(\mathcal{R}_{k-1}(A)) &= \varepsilon_j^{\vee}(\mathcal{R}_{k-1}(A)) + \varepsilon_j(\mathcal{R}_{k-1}(A)) + \operatorname{wt}_j(\mathcal{R}_{k-1}(A)) \\ &= \varepsilon_j^{\vee}(\mathcal{R}(A)) + 1 + \varepsilon_j(\mathcal{R}(A)) + 1 + \operatorname{wt}_j(\mathcal{R}(A)) - 2 \\ &= 0 \end{aligned} \tag{6.16}$$

For $\mathcal{R}_{k-2}(A)$,

$$\varepsilon_j^{\vee}(\mathcal{R}_{k-2}(A)) = \delta_{j,k-2} + \delta_{j,-1} = \delta_{j,-1} = \varepsilon_j^{\vee}(\mathcal{R}(A)). \tag{6.17}$$

Also

$$\varepsilon_j(\mathcal{R}_{k-2}(A)) = \varepsilon_j(\widetilde{f}_{j-1}^{\vee}\mathcal{R}_{k-1}(A)) = \varepsilon_j(\mathcal{R}_{k-1}(A)) \tag{6.18}$$

by Remark 4.12.

If $k-2 \neq j+1$ (i.e. $\ell \neq 2$) then

$$\begin{aligned} \operatorname{jump}_j(\mathcal{R}_{k-2}(A)) &= \varepsilon_j^{\vee}(\mathcal{R}(A)) + \varepsilon_j(\mathcal{R}(A)) + 1 + \operatorname{wt}_j(\mathcal{R}(A)) - 2 + 1 \\ &= 0. \end{aligned} \tag{6.19}$$

Since $\varepsilon_j(\widetilde{f}_i^\vee B) = \varepsilon_j(B)$, $\varepsilon_j^\vee(\widetilde{f}_i^\vee B) = \varepsilon_j^\vee(B)$, and $\mathrm{wt}_j(\widetilde{f}_i^\vee B) = \mathrm{wt}_j(B)$ when $i \notin \{j-1, j, j+1\}$, similar computations show $\mathrm{jump}_j(\mathcal{R}_t(A)) = 0$ for $k - \ell < t \leq k$.

If $k - \ell > 0$, we check

$$\begin{aligned} \mathrm{jump}_j(\mathcal{R}_{k-\ell}(A)) &= \delta_{j,j+1} + \delta_{j,-1} + \varepsilon_j(\widetilde{f}_{j+1}^\vee \mathcal{R}_{k-\ell+1}(A)) + \mathrm{wt}_j(\widetilde{f}_{j+1}^\vee \mathcal{R}_{k-\ell+1}(A)) \\ &= \mathrm{jump}_j(\mathcal{R}_{k-\ell+1}(A)) + 1 = 1. \end{aligned} \tag{6.20}$$

(Note that when $\ell = 2$, $\varepsilon_j^\vee(\mathcal{R}_{k-\ell}(A)) = \varepsilon_j^\vee(\mathcal{R}_{k-\ell+1}(A)) - 1$ but $\mathrm{wt}_j(\widetilde{f}_{j+1}^\vee \mathcal{R}_{k-\ell+1}(A)) = \mathrm{wt}_j(\mathcal{R}_{k-\ell+1}(A)) + 2$ so the equality still holds.)

Next $\mathrm{jump}_j(\mathcal{R}_{k-\ell-1}(A)) = \mathrm{jump}_j(\widetilde{f}_j^\vee \mathcal{R}_{k-\ell}(A)) = 1 - 1 = 0$. Now all other inductive computations for $\mathrm{jump}_j(\mathcal{R}_t(A))$ are identical to the above computations, down to $t = 0$, for which $\mathcal{R}_0(A) = A$. Here if $0 = t \equiv j + 1 \bmod \ell$ then because $\varepsilon_j^\vee(A) = 0 \neq \delta_{j,-1} + \delta_{j,0}$ we instead get $\mathrm{jump}_j(A) = 0$.

The second case, $k = j+1$ but $J > 0$, is also very similar to the above. The only difference is that $\mathrm{jump}_j(\mathcal{R}(A)) \neq 0$ hence $\varepsilon_j(\mathcal{R}_{k-1}(A)) = \varepsilon_j(\widetilde{f}_j^\vee \mathcal{R}(A)) = \varepsilon_j(\mathcal{R}(A))$. Regardless $\mathrm{jump}_j(\mathcal{R}_{k-1}(A)) = \mathrm{jump}_j(\widetilde{f}_j^\vee \mathcal{R}(A)) = J - 1$. We check

$$\begin{aligned} \mathrm{jump}_j(\mathcal{R}_{k-2}(A)) &= \delta_{j,k-2} + \delta_{j,-1} + \varepsilon_j(\widetilde{f}_{j-1}^\vee \mathcal{R}_{k-1}(A)) + \mathrm{wt}_j(\widetilde{f}_{j-1}^\vee \widetilde{f}_j^\vee \mathcal{R}(A)) \\ &= 0 + \varepsilon_j^\vee(\mathcal{R}(A)) + \varepsilon_j(\mathcal{R}(A)) + \mathrm{wt}_j(\mathcal{R}(A)) - 2 - \langle h_j, \alpha_{j-1} \rangle \\ &= \begin{cases} J - 1 & \text{if } \ell \neq 2 \\ J & \text{if } \ell = 2. \end{cases} \end{aligned}$$

Note in the case $\ell = 2$ that $k - 2 \equiv j + 1$, so this is consistent with the statement of the lemma. The rest of the proof is identical to that in Case 1.

Finally we consider $k \not\equiv j+1 \bmod \ell$. Letting $j_0 \in \mathbb{Z}$, $j_0 < k$ be maximal such that $j_0 \equiv j \bmod \ell$, it is clear that $\mathrm{jump}_j(\mathcal{R}_t(A)) = \mathrm{jump}_j(\mathcal{R}(A))$ for all $t > j_0 + 1$.

Then

$$\mathrm{jump}_j(\mathcal{R}_{j_0+1}(A)) = \delta_{j,j_0+1} + \delta_{j,-1} + \varepsilon_j(\widetilde{f}_{j+1}^\vee \mathcal{R}_{j_0+2}(A)) + \mathrm{wt}_j(\widetilde{f}_{j+1}^\vee \mathcal{R}_{j_0+2}(A)) \tag{6.21}$$
$$= 0 + \delta_{j,-1} + \varepsilon_j(\mathcal{R}_{j_0+2}(A)) + \mathrm{wt}_j(\mathcal{R}_{j_0+2}(A)) - \langle h_j, \alpha_{j+1} \rangle \tag{6.22}$$
$$= \mathrm{jump}_j(\mathcal{R}_{j_0+2}(A)) + 1 = J + 1. \tag{6.23}$$

In the case $\ell \neq 2$ this follows as $\delta_{j,-1} = \varepsilon_j^\vee(\mathcal{R}_{j_0+2}(A))$ and $\langle h_j, \alpha_{j+1} \rangle = -1$. In the case $\ell = 2$, we have $\varepsilon_j^\vee(\mathcal{R}_{j_0+2}(A)) = 1 + \delta_{j,-1}$ but $\langle h_j, \alpha_{i+1} \rangle = -2$.

Next $\mathrm{jump}_j(\mathcal{R}_{j_0}(A)) = \mathrm{jump}_j(\widetilde{f}_j^\vee \mathcal{R}_{j_0+1}(A)) = \mathrm{jump}_j(\mathcal{R}_{j_0+1}(A)) - 1 = J$.

Also for $\ell \neq 2$,

$$\mathrm{jump}_j(\mathcal{R}_{j_0-1}(A)) = \delta_{j,j-1} + \delta_{j,-1} + \varepsilon_j(\widetilde{f}_{j-1}^\vee \mathcal{R}_{j_0}(A)) + \mathrm{wt}_j(\widetilde{f}_{j-1}^\vee \mathcal{R}_{j_0}(A)) \tag{6.24}$$
$$= \mathrm{jump}_j(\mathcal{R}_{j_0}(A)) = J \tag{6.25}$$

as $\varepsilon_j^\vee(\mathcal{R}_{j_0}(A)) = 1 + \delta_{j,-1}$ and $\langle h_j, \alpha_{j-1} \rangle = -1$. We don't consider $\ell = 2$ as $j_0 - 1 \equiv j_0 + 1$ and we have already considered that case. We note that the calculations of $\mathrm{jump}_j(\mathcal{R}_t(A))$ only depend on $t \bmod \ell$ and so we are done. □

Proof of Theorem 6.5. For ease of exposition we set $i = 0$ in the proof. In fact we will prove a slightly stronger statement, that when $\varepsilon_j(T_{0;r(A)}) < \varphi_j^{\Lambda_{-1}}(\mathcal{R}(A))$ and $\mathrm{pr}_{\Lambda_0} \widetilde{f}_j A \neq \mathbf{0}$ then $r(\widetilde{f}_j A) = r(A)$ and for $0 < t \leq r(A)$, $\mathcal{R}_t(\widetilde{f}_j A) = \widetilde{f}_j \mathcal{R}_t(A)$.

First note that in the case $\mathcal{R}(A) = \mathbb{1}$ the theorem is obvious as $\widetilde{f}_j \mathcal{R}(A) = L(j)$ and $\widetilde{f}_j T_{0;k} = \operatorname{cosoc} \operatorname{Ind} T_{0;k} \boxtimes L(j)$. So from now on assume $\mathcal{R}(A) \neq \mathbb{1}$.

Case 1: Suppose $\varepsilon_j(T_{0;k}) = 0$. In particular, $j \neq k - 1$.

- *Case 1a*: Suppose $\mathrm{jump}_j(\mathcal{R}(A)) = 0$. By Lemma 6.8,

$$\mathrm{jump}_j(A) = \begin{cases} 0 & 0 \not\equiv j+1 \bmod \ell, \\ 1 & 0 \equiv j+1 \bmod \ell. \end{cases} \tag{6.26}$$

So when $j \neq -1$ as $\mathrm{jump}_j(A) = 0$, $\mathrm{pr}_{\Lambda_0} \widetilde{f}_j A = \mathbf{0}$ and so we need not consider this case. Hence we may assume $j = -1$. Further, as $\mathrm{jump}_{-1}(\mathcal{R}(A)) = 0$ and $\varepsilon^\vee_{-1}(\mathcal{R}(A)) = 1$ we have

$$\operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes \mathcal{R}(A) \xrightarrow{\cong} \operatorname{Ind} T_{0;k} \boxtimes \mathcal{R}(A) \boxtimes L(-1) \tag{6.27}$$

$$\twoheadrightarrow \operatorname{Ind} A \boxtimes L(-1) \twoheadrightarrow \widetilde{f}_{-1} A \tag{6.28}$$

This implies $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T_{0;k} \boxtimes L(-1) \boxtimes L(-1) \neq \mathbf{0}$, forcing $k \equiv -1 \bmod \ell$ by Propositions 5.4 and 5.5. But then we have

$$\operatorname{Ind} T(0, \ldots, -2) \boxtimes L(-1) \boxtimes \mathcal{R}(A) \twoheadrightarrow \widetilde{f}_{-1} A. \tag{6.29}$$

Because

$$\operatorname{Ind} T(0, \ldots, -2) \boxtimes \mathcal{R}(A) \twoheadrightarrow A \tag{6.30}$$

we see $\varepsilon_{-1}(\mathcal{R}(A)) = \varepsilon_{-1}(A) = \varepsilon_{-1}(\widetilde{f}_{-1} A) - 1$. If we had

$$\operatorname{Ind} N \boxtimes \mathcal{R}(A) \twoheadrightarrow \widetilde{f}_{-1} A \tag{6.31}$$

for any composition factor N of $\operatorname{Ind} T_{0;k} \boxtimes L(-1)$ other than $\widetilde{f}_{-1} T_{0;k}$, the Shuffle Lemma would yield $\varepsilon_{-1}(\widetilde{f}_{-1} A) = \varepsilon_{-1}(\mathcal{R}(A))$, a contradiction. Hence we have

$$\operatorname{Ind} T_{0;k+1} \boxtimes \mathcal{R}(A) = \operatorname{Ind} \widetilde{f}_{-1} T_{0;k} \boxtimes \mathcal{R}(A) \twoheadrightarrow \widetilde{f}_{-1} A. \tag{6.32}$$

- *Case 1b*: Suppose that $\mathrm{jump}_j(\mathcal{R}(A)) = J > 0$. Again by Lemma 6.8

$$\mathrm{jump}_j(A) = \begin{cases} J & 0 \not\equiv j+1 \bmod \ell, \\ J+1 & 0 \equiv j+1 \not\equiv k \bmod \ell. \end{cases} \tag{6.33}$$

Note $\widetilde{f}_j \mathcal{R}(A) \in \mathrm{Rep}^{\Lambda_{-1}}$ as $\mathrm{jump}_j(\mathcal{R}(A)) > 0$.

We compute

$$\widetilde{f}_j \mathcal{R}_{k-1}(A) = \widetilde{f}_j \widetilde{f}^\vee_{k-1} \mathcal{R}(A) = \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A) \tag{6.34}$$

as $j \neq k-1$ as in Case 1. Also, clearly $\widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A) \in \mathrm{Rep}^{\Lambda_{k-1}+\Lambda_{-1}}$ and $\widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A) \notin \mathrm{Rep}^{\Lambda_{-1}}$. Assume we have shown

$$\widetilde{f}_j \mathcal{R}_t(A) = \widetilde{f}^\vee_t \ldots \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A) \in \mathrm{Rep}^{\Lambda_t + \Lambda_{-1}} \setminus \mathrm{Rep}^{\Lambda_{-1}}. \tag{6.35}$$

Then we compute $\widetilde{f}_j \mathcal{R}_{t-1}(A) = \widetilde{f}_j \widetilde{f}^\vee_{t-1} \mathcal{R}_t(A)$. If $t-1 \not\equiv j \bmod \ell$ this is equal to $\widetilde{f}^\vee_{t-1} \widetilde{f}_j \mathcal{R}_t(A) = \widetilde{f}^\vee_{t-1} \ldots \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A)$ by the inductive hypothesis. If instead $t \equiv j+1 \bmod \ell$ then

$$\mathrm{jump}_j(\mathcal{R}_t(A)) = J + 1 > 1. \tag{6.36}$$

So by Lemma 4.11.2

$$\widetilde{f}_j \mathcal{R}_{t-1}(A) = \widetilde{f}_j \widetilde{f}^\vee_j \mathcal{R}_t(A) = \widetilde{f}^\vee_j \widetilde{f}_j \mathcal{R}_t(A) = \widetilde{f}^\vee_j \widetilde{f}^\vee_t \ldots \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A) \tag{6.37}$$

$$= \widetilde{f}^\vee_{t-1} \ldots \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A). \tag{6.38}$$

By downwards induction $\widetilde{f}_j A = \widetilde{f}_j \mathcal{R}_0(A) = \widetilde{f}^\vee_0 \ldots \widetilde{f}^\vee_{k-1} \widetilde{f}_j \mathcal{R}(A)$. Certainly $\widetilde{f}_j \mathcal{R}_{t-1}(A) \in \mathrm{Rep}^{\Lambda_{t-1}+\Lambda_{-1}} \setminus \mathrm{Rep}^{\Lambda_{-1}}$ as $\mathrm{jump}_j(\mathcal{R}_{t-1}(A)) > 0$ and $\mathcal{R}_{t-1}(A) \in \mathrm{Rep}^{\Lambda_{t-1}+\Lambda_{-1}} \setminus \mathrm{Rep}^{\Lambda_{-1}}$ when $t-1 > 0$. And we already know $\widetilde{f}_j A \in \mathrm{Rep}^{\Lambda_0}$. This completes the induction and furthermore shows

$$r(\widetilde{f}_j A) = r(A), \qquad \mathcal{R}(\widetilde{f}_j A) = \widetilde{f}_j \mathcal{R}(A) \tag{6.39}$$

as well as the stronger statement $\mathcal{R}_t(\widetilde{f}_j A) = \widetilde{f}_j \mathcal{R}_t(A)$ for $0 \le t \le k$. In other words we have

$$\operatorname{Ind} T_{0;k} \boxtimes \widetilde{f}_j \mathcal{R}(A) \to \widetilde{f}_j A. \tag{6.40}$$

Case 2: Suppose that $\varepsilon_j(T_{0;k}) = 1$. This is the only other possibility as $\varepsilon_j(T_{0;k}) \le 1$ for all $j \in I$. Note $k \equiv j+1 \bmod \ell$.

- *Case 2a*: If $\varphi_j^{\Lambda_{-1}}(\mathcal{R}(A)) = 0$ then $\mathrm{jump}_j(A) = 0$ by Lemma 6.8 so $\mathrm{pr}_{\Lambda_0} \widetilde{f}_j A = 0$ and we need not consider this case.
- *Case 2b*: If $\varphi_j^{\Lambda_{-1}}(\mathcal{R}(A)) = 1$ then again by Lemma 6.8 as $\mathcal{R}(A) = \mathcal{R}_k(A)$ and $k \equiv j+1 \bmod \ell$

$$\mathrm{jump}_j(A) = \mathrm{jump}_j(\mathcal{R}_0(A)) = \begin{cases} 0 & 0 \not\equiv j+1 \bmod \ell \\ 1 & 0 \equiv j+1 \bmod \ell. \end{cases} \tag{6.41}$$

So we need not consider this case unless $j = -1$. However we will show, this case cannot arise as we assumed $\mathcal{R}(A) \ne \mathbb{1}$. Note $\mathrm{jump}_j(\mathcal{R}_{k-1}(A)) = \mathrm{jump}_{-1}(\widetilde{f}_{-1}^{\vee}\mathcal{R}(A)) = 0$ and $\mathcal{R}_{k-1}(A) \in \mathrm{Rep}^{2\Lambda_{-1}}$. Thus we have

$$\operatorname{Ind} T_{0;k-1} \boxtimes L(-1) \boxtimes \mathcal{R}_{k-1}(A) \xrightarrow{\cong} \operatorname{Ind} T_{0;k-1} \boxtimes \mathcal{R}_{k-1}(A) \boxtimes L(-1) \tag{6.42}$$

$$\twoheadrightarrow \operatorname{Ind} A \boxtimes L(-1) \twoheadrightarrow \widetilde{f}_{-1} A. \tag{6.43}$$

However $\mathrm{pr}_{\Lambda_0} \operatorname{Ind} T(0,\dots,k-1) \boxtimes L(-1) \boxtimes L(-1) \boxtimes L(-1) = \mathbf{0}$ by Proposition 5.6.4b, which is a contradiction to $\mathrm{pr}_{\Lambda_0} \widetilde{f}_{-1} A \ne \mathbf{0}$.

- *Case 2c*: When $\varphi_j^{\Lambda_{-1}}(\mathcal{R}(A)) = J > 1$ the argument is similar to Case 1*b*.

□

6.3. **Other types.** We can repeat the arguments above to prove similar theorems in affine type B, C, and D, using the Kirillov-Reshetikhin crystal $B^{1,1}$ of appropriate type, and the modules corresponding to the nodes studied in [27] in place of $T_{i;k}$. This is work in progress [20].

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